Week 4 - Lecture 7

January 27^{th}, 2020

$\frac{1}{2} \times \frac{1}{2}$

$\frac{1}{4}$

$\frac{1}{2} \times \frac{2}{1}$

Ex: let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation

$T\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} a-b \\ b-a \end{pmatrix}$ } order is $\begin{pmatrix} a \\ b \end{pmatrix}$ so switch a & b $\begin{pmatrix} a-b \\ -a\ b \end{pmatrix}$

$\begin{pmatrix} a-b \\ -a+b \end{pmatrix}$

a) $T = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

b) Find matrix A corresponding to T ($R^2 - R^2$ should be 2x2)

$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

c) use A to Find

$T\begin{pmatrix} 0 \\ -3 \end{pmatrix}, T\begin{pmatrix} 2 \\ 0 \end{pmatrix}, T\begin{pmatrix} 1 \\ -2 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \end{pmatrix}, T\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$V_1 \to T\left(\begin{pmatrix} 0 \\ -3 \end{pmatrix}\right) \overset{\text{same as}}{=} A\begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{pmatrix} (1\times 0) + (-1 \times -3) = (0+3) = \\ (-1\times 0) + (1 \times -3) = (0-3) = \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

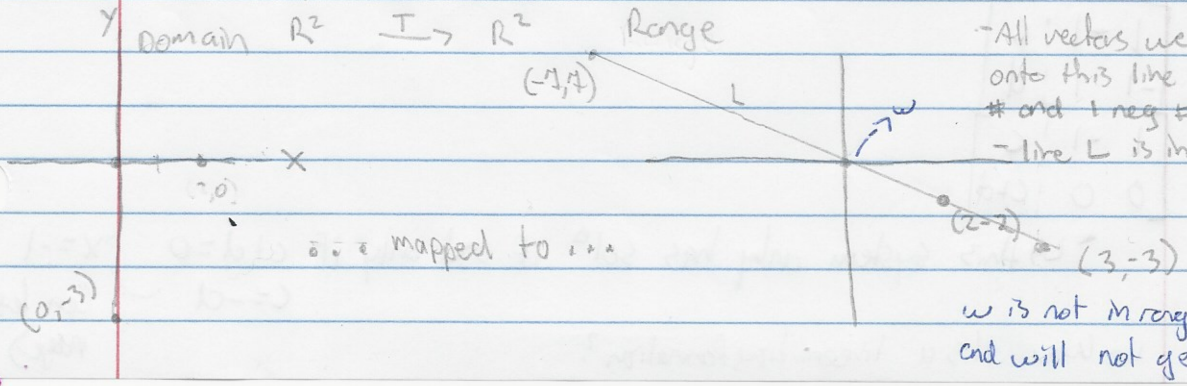
$V_2 \to T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = A\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$V_3 \to T\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ difference between multiplying and matrizes

$V_4 \to T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$V_5 \to T\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \end{pmatrix}$

linear transformation that we have is



Corresponding Matrix

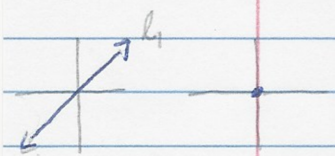
- All vectors in this line are in range
- All vectors we chose will map onto this line and are 1 pos # and 1 neg #
- line L is in range of T vectors

w is not in range of T of A() and will not get mapped what is Kernel

$Ker(T) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} a-b \\ b-a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

↳ set of all a, and b's such that $a-b=0$, $b-a=0$, means all a and b's such that $a=b$



↳ any vector on this line is mapped to origin

$= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; a=b \right\}$ l_1 if $a=b=0$ then $a-b=0$, $b-a=0$

Use matrices to solve Kernel, solve homogeneous matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc:c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_1+R_2 \end{array}$$

$$\left[\begin{array}{cc:c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

pivot free

$x - y = 0 \rightarrow x = y$

$y = 0 \rightarrow y = y$

if we want to check if vector $v = \begin{pmatrix} c \\ d \end{pmatrix}$ is in the range of T, we need to check if $[A|v]$ has any solution

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$$

↳ which vectors have sol^n in range

$$\left[\begin{array}{cc:c} 1 & -1 & c \\ -1 & 1 & d \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & c \\ 0 & 0 & c+d \end{array}\right]$$

↳ this system only has sol^n if and only if $c+d=0$, $c=-d$ } $x=-1$ (on last page)

we know its a linear transformation?

-one to one is same as onto
for square matrices

Def: let $T: \mathbb{R}^n - \mathbb{R}^m$ (from) be a linear transformation

① if for every vector $w \in \mathbb{R}^m$ there exists at least $v \in \mathbb{R}^n$ such that $T(v) = w$

$\therefore$ we say T is "onto"

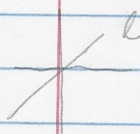
-no vector is mapped to 0 except 0 vector

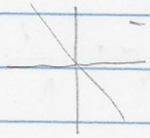
-image & range are basically equivalent

② if for every vector $w \in \mathbb{R}^m$ there exists at most one $v \in \mathbb{R}^n$ such that $T(v) = w$

$\therefore$ we say T is "one-to-one"

we say its "onto" if it covers everything here





-this is not onto

$$\left.\begin{aligned} &V_1, V_3 \text{ sth} \\ &T(V_1) = w \\ &T(V_3) = w \end{aligned}\right\} \text{this ex is not 1:1}$$

This particular vector is map of Z in domain

-image is always sitting inside range

Ex: let $T: \mathbb{R}^2 - \mathbb{R}^2$ be given by

$$T(e_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\quad T(e_2) = \begin{pmatrix} 2 \\ 4 \end{pmatrix},\quad T(e_3) = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

- given linear trasformation depends on

a) is T onto

b) is T 1:1

- trasformation is defined by 3 vectors

Matrix

$$A = \{T(e_1), T(e_2), T(e_3)\} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

RREF

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

A) Not "onto" becase second row does not have leading 1

↳ if we want to be onto, each row has to have leading 1

b) No if it wants to be 1:1 each column has to have leading 1

Next Example

① check if the linear trasformation $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ x-y \\ y+z \\ x-y \end{bmatrix}$ is

$$T: \mathbb{R}^3 - \mathbb{R}^4$$

① onto?

② one-to-one?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \to \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 - R_1 \\ R_3 \\ R_4 - R_1 \end{array} \to \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{array}\right]\begin{array}{l} R_1 \\ -\frac{1}{2}R_2 \\ R_2 \\ R_3 \end{array}$$

$$\to \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 - R_2 \\ R_4 + 2R_2 \end{array} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

→ is it onto? No becase last row does not have a leading 1

→ is it 1:1? Yes, every column has a leading 1

Check if $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a + 2b + 3c \\ a - b + c \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2/3 \end{bmatrix} \begin{matrix} R_1 \\ -1/3 R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 5/3 \\ 0 & 1 & 2/3 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \end{matrix}$$

① onto? yes, each row has a leading 1

② 1:1? No, the third column does not have a leading 1

Note: let $T: \mathbb{R}^n \to \mathbb{R}^m$ — n & m refer to plane x, y, z

if $n < m$ T can not be onto

if $n > m$ T can not be 1:1

↳ something big to something smaller have to be mapped to same vector

in terms of matrix form

$$A = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \\ 0 & 0 & \\ 0 & 0 & \end{bmatrix}_{m \times n}$$ m = rows, n = colums

if $n < m$, for sure there are rows (at least 1) without a leading 1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

if $n > m$, for sure there are columns (at least 1) without a leading 1

let $T\ \mathbb{R}^n - \mathbb{R}^m$ be a linear transformation then the following are equivalent

1) T is one-to-one if and only if $T(x) = 0$, has only trivial sol^n

Kernel (T) = 0

- each column

- each column of RREF has leading 1

T is onto if and only if - range of (-T) is everything $= \mathbb{R}^m$

- $Ax = b$ has sol^n(s) for every $b = \mathbb{R}^m$
- each row of RREF has leading 1

- if we can find any random vector in image
we can find mapped vector
means always sol^n