MATH 1009 **TEST 3 - SOLUTIONS** November 2019

Based on sections 4.1 - 4.5.

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1. [**9 Marks**] Find the intervals where each function f is increasing and those where it is decreasing:

[3] **(a)** $f(x) = e^{3-x}$.

[6] **(b)** $f(x) = x^3 - 6x^2$.

Solution:

(a) $f'(x) = e^{3-x} \cdot (3-x)' = -e^{3-x} < 0$ for all real x. Thus, by the First Derivative Test, the function is decreasing for $x \in \mathbb{R}$.

(b) $f'(x) = 3x^2 - 12x = 3x(x-4)$. $f'(x) > 0$ when $x \in (-\infty, 0)$ or when $x \in (4, \infty)$, so f is increasing on those intervals. $f'(x) < 0$ when $x \in (0, 4)$, so f is decreasing there.

2. [**12 Marks**] The quantity demanded each week x is related to the unit price p by the demand equation

$$x = f(p) = \sqrt{400 - 5p} = (400 - 5p)^{1/2} \qquad (0 < p < 80).$$

The elasticity of demand is given by the formula $E(p) = -\dfrac{p\,f'(p)}{f(p)}$.

[9] **(a)** Is the demand elastic or inelastic at $p = 40$? at $p = 60$?
[2] **(b)** For what price p is the demand unitary?
[1] **(c)** If the unit price is increased slightly from 40, will the revenue increase or decrease?

Solution:

(a) We compute

$$f'(p) = \left(\sqrt{400 - 5p}\right)' = \frac{1}{2} \cdot \left(\sqrt{400 - 5p}\right)^{-\frac{1}{2}} \cdot (-5) = -\frac{5}{2\sqrt{400 - 5p}}.$$

Thus, $E(p) = -\dfrac{p \cdot f'(p)}{f(p)} = -\dfrac{p \cdot \dfrac{-5}{\sqrt{400-5p}}}{2\sqrt{400 - 5p}} = \dfrac{5p}{2(400 - 5p)}$.

$E(40) = \dfrac{5 \cdot 40}{2(400 - 5 \cdot 40)} = \dfrac{1}{2} < 1$, so the demand is inelastic when $p = 40$.

$E(60) = \dfrac{5 \cdot 60}{2(400 - 5 \cdot 60)} = \dfrac{3}{2} > 1$, so the demand is elastic when $p = 60$.

(b) $E(p) = \dfrac{5p}{2(400 - 5p)} = 1$ when $5p = 2(400 - 5p)$, $5p + 10p = 800$, so for $p = 800/15 \approx 53.3$ the demand is unitary.

(c) Since the demand is inelastic when $p = 40$, the revenue will increase if the unit price is increased slightly from 40.

3. **[19 Marks]** Find and classify the critical numbers of the following functions:

[6] (a) $f(x) = -2x^3 + 6x + 23$. [7] (b) $g(x) = \dfrac{2x^2}{x-3}$. [6] (c) $f(x) = \log_3(5x^2 + 4)$.

Solution:

(a) $f(x)$ is a polynomial function, so its domain is the set of all real numbers. The only source of critical numbers is the condition $f'(x) = 0$.

$$f'(x) = -6x^2 + 6 = -6(x^2 - 1) = -6(x-1)(x+1) = 0 \text{ when } x = 1 \text{ or } x = -1.$$

As $f'(x) > 0$ for $-1 < x < 1$ and $f'(x) < 0$ otherwise, by the First Derivative Test, at $x = -1$ the function has a local minimum and at $x = 1$ - a local maximum.

(b) $g(x)$ is a rational function, defined on the set of all real numbers for which $x - 3 \neq 0$, that is, $x \neq 3$.

$$g'(x) = \frac{4x(x-3) - 2x^2(1)}{(x-3)^2} = \frac{2x^2 - 12x}{(x-3)^2} = \frac{2x(x-6)}{(x-3)^2}.$$

The derivative $g'(x)$ is not defined for $x = 3$. However, since $x = 3$ is not in the domain of g, the number $x = 3$ is not a critical number of g.

$$g'(x) = 0 \text{ when } x = 0 \text{ or } x = 6.$$

As $f'(x) < 0$ for $0 < x < 6$ and $f'(x) > 0$ otherwise, by the First Derivative Test, at $x = 0$ the function has a local maximum and at $x = 6$ - a local minimum.

(c) The domain of f is the set of all real numbers, as $5x^2 + 4 > 0$ for all real x.

$f'(x) = \dfrac{1}{(5x^2+4)\ln 3} \cdot (5x^2+4)' = \dfrac{10x}{(5x^2+4)\ln 3} = 0$ when $x = 0$. The derivative is defined for all real x. Thus, the function has one critical number $x = 0$.

$f'(x) > 0$ when $x > 0$ and $f'(x) < 0$ when $x < 0$, which means that the derivative is changing its sign at $x = 0$ from "-" to "+", and therefore, the function has a local minimum at $x = 0$.