

Based on Chapter 2 (Limits) + Chapter 3 (Derivative and Rules of Differentiation, Implicit Differentiation, up to higher order derivatives.)

PART A: MULTIPLE CHOICE ANSWERS: a, b, b, d, d.

[5 × 2 marks each = 10 marks]

A1. $\lim_{x \rightarrow 0} \frac{3x^5 - 2x^3 + 7x + 1}{x^5 + 5}$ is

- (a) $\frac{1}{5}$ (b) 3 (c) 0 (d) ∞ (e) None of the above.

A2. $\lim_{x \rightarrow 3} \frac{9 - x^2}{x - 3}$ is

- (a) 6 (b) -6 (c) 0 (d) ∞ (e) None of the above.

A3. $\lim_{x \rightarrow -\infty} \frac{7x^4 + x^3 + 3x^2 - 3}{x^7 + 3x^2 + 1}$ is

- (a) 7 (b) 0 (c) -3 (d) ∞ (e) None of the above.

A4. $\lim_{x \rightarrow \infty} \frac{3x^5 + x^3 - 4x^2 - 11}{x^2 + x + 1}$ is

- (a) 3 (b) 0 (c) -11 (d) ∞ (e) None of the above.

A5. The function $g(x) = \frac{x - 3}{x^4 + 1}$ is discontinuous at

- (a) $x = \pm 1$ (b) $x = 3$ (c) $x = -1$ (d) continuous everywhere

- (e) None of the above.

PART B: LONG STYLE QUESTIONS.

[7 marks] **B1.** Let $f(x) = \begin{cases} 3, & x < 1, \\ -x, & x \geq 1. \end{cases}$

[1] (a) Sketch the graph of f .

[5] (b) Find the following limits:

(i) $\lim_{x \rightarrow 1^+} f(x) = -1;$

(ii) $\lim_{x \rightarrow 1^-} f(x) = 3;$

(iii) $\lim_{x \rightarrow 1} f(x) = \text{DNE (Does Not Exist)};$

(iv) $\lim_{x \rightarrow -15} f(x) = 3;$

(v) $\lim_{x \rightarrow 6} f(x) = -6;$

[1] (c) Is the function f continuous at $x = 1$? Explain.

Since the left-hand-side limit (i) is not the same as the right-hand-side limit (ii), the limit does not exist at $x = 1$. Thus, the function is discontinuous at $x = 1$. The graph has a gap at $x = 1$.

[5 marks] **B2.** Use implicit differentiation to find $\frac{dy}{dx}$ if $2x^3 + 3xy - y^2 = 5$.

Solution:

Compute the derivative with respect to x of both sides of the equation :

$$6x^2 + 3 \cdot 1 \cdot y + 3x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = -\frac{6x^2 + 3y}{3x - 2y}.$$

[8 marks] B3. Use the Intermediate Value Theorem to show that the equation has at least one solution in the designated interval. Explain why the theorem is applicable.

[4] (a) $x^3 + 2x^2 - 5 = 0$, in $(1, 2)$

[4] (b) $4x - 1 = 2^x$, in $(0, 1)$.

Solution: (a) $f(x) = x^3 + 2x^2 - 5$ is a polynomial, therefore, it is continuous on the interval $[1, 2]$. Thus, we can apply the IVT on that interval.

$$f(1) = (1)^3 + 2 \cdot 1^2 - 5 = -2 < 0; \quad f(2) = 2^3 + 2 \cdot 2^2 - 5 = 8 + 8 - 5 = 11 > 0.$$

Since the values of the function have opposite signs at the endpoints of the interval, then there is at least one point on the interval, at which this function assumes the value of zero.

(b) In the context of this problem, $f(x) = 4x - 1 - 2^x$. As a difference of two continuous function, it is continuous on the interval $[0, 1]$,

$$f(0) = 4 \cdot 0 - 1 - 2^0 = -2 < 0; \quad f(1) = 4 \cdot 1 - 1 - 2^1 = 4 - 3 = 1 > 0.$$

Since the values of the function have opposite signs at the endpoints of the interval, then there is at least one point on the interval, at which this function assumes the value of zero.

[10 marks] B4. Find the derivative of the following functions using the appropriate rules of differentiation:

[3] (a) $f(x) = \frac{4x^2}{2x^3 - 7}$. (Simplify the numerator of the resulting expression.)

[2] (b) $g(x) = 3^{1-4x}$.

[2] (c) $h(x) = 4(x^5 + 2x^2 - 1)^{1/4}$.

[3] (d) $k(x) = \log_5(2x^3 + e^x)$.

Solution:

$$(a) \quad f'(x) = \frac{(4x^2)'(2x^3 - 7) - 4x^2(2x^3 - 7)'}{(2x^3 - 7)^2} = \frac{8x(2x^3 - 7) - 4x^2 \cdot 6x^2}{(2x^3 - 7)^2} = \frac{-8x^4 - 56x}{(2x^3 - 7)^2}.$$

$$(b) \quad g'(x) = 3^{1-4x} \cdot \ln 3 \cdot (1 - 4x)' = (-4 \ln 3) 3^{1-4x}.$$

$$(c) \quad h'(x) = 4 \cdot \frac{1}{4}(x^5 + 2x^2 - 1)^{-3/4} \cdot (x^5 + 2x^2 - 1)' = (x^5 + 2x^2 - 1)^{-3/4}(5x^4 + 4x).$$

$$(d) \quad k'(x) = \frac{1}{(2x^3 + e^x) \ln 5} \cdot (2x^3 + e^x)' = \frac{6x^2 + e^x}{(2x^3 + e^x) \ln 5}.$$