MAT3379 (Winter 2020)
Assignment 1 - Solutions

Q1. (8 points) Let $\{Z_t\}$ be independent normal random variables with mean 0 and variance σ^2. Let a, b, c be constants. Which of the following processes are stationary? Evaluate mean and autocovariance function.

(a) $X_t = Z_t \cos(at) + Z_{t-1} \sin(bt)$.

(b) $X_t = a + bZ_t + cZ_{t-2}$.

(c) $X_t = Z_t Z_{t-1}$.

Solution to Q1:

Note that $\mathrm{E}[Z_t Z_{t+h}] = \mathrm{E}[Z_t]\mathrm{E}[Z_{t+h}] = 0$ for all $h \neq 0$ and

(a) We have

$$\mathrm{E}[X_t] = \cos(at)\mathrm{E}[Z_t] + \sin(bt)\mathrm{E}[Z_{t-1}] = 0$$

since $\mathrm{E}[Z_t] = 0$. Hence, the mean does not depend on t.
Also,

$$\begin{aligned}\gamma_X(t, t+h) = \mathrm{E}[X_t X_{t+h}] =& \cos(at)\cos(a(t+h))\mathrm{E}[Z_t Z_{t+h}] + \sin(bt)\sin(b(t+h))\mathrm{E}[Z_{t-1}Z_{t+h-1}] + \\ &+ \cos(at)\sin(b(t+h))\mathrm{E}[Z_t Z_{t+h-1}] + \sin(bt)\cos(a(t+h))\mathrm{E}[Z_{t-1}Z_{t+h}].\end{aligned}$$

For $h = 0$ we get

$$\begin{aligned}\gamma_X(t,t) = \mathrm{E}[X_t^2] =& \cos^2(at)\mathrm{E}[Z_t^2] + \sin^2(bt)\mathrm{E}[Z_{t-1}^2] + \\ &+ \cos(at)\sin(bt)\mathrm{E}[Z_t]\mathrm{E}[Z_{t-1}] + \sin(bt)\cos(at)\mathrm{E}[Z_{t-1}]\mathrm{E}[Z_t] \\ =& \cos^2(at)\sigma^2 + \sin^2(bt)\sigma^2 + 0 + 0.\end{aligned}$$

Hence, the variance depends on t - the sequence is not stationary.
Marking: This part will not be marked.

(b) We have

$$\mathrm{E}[X_t] = \mathrm{E}[a + bZ_t + cZ_{t-2}] = a$$

since $\mathrm{E}[Z_t] = 0$. Hence, the mean does not depend on t.
Also,

$$\gamma_X(t,t) = \mathrm{Var}[X_t] = \mathrm{Var}(a + bZ_t + cZ_{t-2}) = b^2\sigma_Z^2 + c^2\sigma_Z^2 \quad (1)$$

Hence, the variance does not depend on t. Moreover,

$$\gamma_X(t, t+1) = \mathrm{E}[X_t X_{t+1}] - a^2 = \mathrm{E}\left[(a + bZ_t + cZ_{t-2})(a + bZ_{t+1} + cZ_{t-1})\right] = 0 \quad (2)$$

since none of the indices of Z agree.

$$\gamma_X(t, t+2) = \mathrm{E}[X_t X_{t+1}] - a^2 = \mathrm{E}\left[(a + bZ_t + cZ_{t-2})(a + bZ_{t+2} + cZ_t)\right] = bc\mathrm{E}[Z_t^2] = bc\sigma_Z^2 \quad (3)$$

For $h \geq 2$ we have the same situation as for $\gamma_X(t, t+1)$, the covariance is zero. This, together with (1)-(3) implies that the covariance does not depend on t.

Stationarity follows.
Marking: 1 point for correct computation of the mean; 2 point for correct computation of covariance (you must provide general formula for arbitrary h); 1 point for correct conclusion about stationarity. Total: 4 points.

(c) We have

$$\mathrm{E}[X_t] = \mathrm{E}[Z_t Z_{t-1}] = 0$$

since $\mathrm{E}[Z_t] = 0$. Hence, the mean does not depend on t. Also

$$\gamma_X(t, t+h) = \mathrm{E}[X_t X_{t+h}] = \mathrm{E}[Z_t Z_{t-1} Z_{t+h} Z_{t+h-1}].$$

If $h = 0$, then $\gamma_X(t,t) = \mathrm{E}[Z_t^2]\mathrm{E}[Z_{t-1}^2] = \sigma^2\sigma^2 = \sigma^4$. If $h \geq 1$, then none of the indices $t, t-1, t+h, t+h-1$ agree. Hence, $\gamma_X(t, t+h) = 0$. Hence, the covariance function does not depend on t. Random variables X_t are uncorrelated (but dependent !). The sequence is stationary.

Marking: 1 point for correct computation of the mean; 2 points for correct computation of covariance (you must provide general formula for arbitrary h); 1 point for correct conclusion about stationarity. Total: 4 points.

Q2. Let $\{Z_t\}$ be independent normal random variables with mean 0 and variance $\sigma^2 = 1$. Consider the sequence

$$X_t = Z_t + (Z_{t-1}^2 - 1)\ , t = 1, 2, \ldots\ .$$

- Show that $E[X_t] = 0$.
- Show that $E[X_t X_{t+h}] = 0$ for $h \neq 0$. Hint: Note that $E[Z_t^3] = 0$ since Z_t are normal.

Solution to Q2:

We have

$$E[X_t] = E[Z_t + (Z_{t-1}^2 - 1)] = E[Z_t] + E[Z_{t-1}^2] = 0 + 1 - 1 = 0\ .$$

Next,

$$\begin{aligned}
E[X_t X_{t+1}] &= E[\{Z_t + (Z_{t-1}^2 - 1)\}\{Z_{t+1} + (Z_t^2 - 1)\}] \\
&= E[Z_t Z_{t+1}] + E[Z_t^3] - E[Z_t] \\
&\quad + E[Z_{t-1}^2 Z_{t+1}] + E[Z_{t-1}^2 Z_t^2] - E[Z_{t-1}^2] \\
&\quad - E[Z_{t+1}] - E[Z_t^2] + 1 \\
&= E[Z_t]E[Z_{t+1}] + E[Z_t^3] - E[Z_t] \\
&\quad + E[Z_{t-1}^2]E[Z_{t+1}] + E[Z_{t-1}^2]E[Z_t^2] - E[Z_{t-1}^2] \\
&\quad - E[Z_{t+1}] - E[Z_t^2] + 1 \\
&= 0 + 0 - 0 \\
&\quad + \sigma^2 \times 0 + \sigma^2 \times \sigma^2 - \sigma^2 \\
&\quad - 0 - \sigma^2 + 1\ .
\end{aligned}$$

We use independence, so for example $E[Z_{t-1}^2 Z_t] = E[Z_{t-1}^2]E[Z_t]$.

Marking scheme for Q2:

Marking: This part will not be marked.

Q3. Let $\{Z_t\}$ be independent random variables with mean 0 and variance σ^2. Let $\{Y_t\}$ be a stationary sequence with a covariance function $\gamma_Y(h)$. Assume also that the sequences $\{Z_t\}$ and $\{Y_t\}$ are independent from each other. Define $X_t = Y_t Z_t$.

Verify that for $h \geq 1$ we have $\mathrm{Cov}(X_t, X_{t+h}) = 0$ and $\mathrm{Cov}(X_t^2, X_{t+h}^2) \neq 0$ (at least not for all t and h).

Solution to Q3:

Note that $\mathrm{E}[X_t] = \mathrm{E}[Y_t]\mathrm{E}[Z_t] = 0$. We have for $h \neq 0$,

$$\gamma_X(t, t+h) = \mathrm{Cov}(X_t, X_{t+h}) = \mathrm{E}[X_t X_{t+h}] = \mathrm{E}[Z_t]\mathrm{E}[Z_{t+h}]\mathrm{E}[Y_t Y_{t+h}] = 0.$$

On the other hand for $h \neq 0$,

$$\begin{aligned}\gamma_{X^2}(t,t+h) =& \mathrm{Cov}(X_t^2, X_{t+h}^2) = \mathrm{E}[X_t^2 X_{t+h}^2] - \mathrm{E}[X_t^2]\mathrm{E}[X_{t+h}^2] \\ =& \mathrm{E}[Y_t^2 Y_{t+h}^2]\mathrm{E}[Z_t^2]\mathrm{E}[Z_{t+h}^2] - \mathrm{E}[Y_t^2]\mathrm{E}[Y_{t+h}^2]\mathrm{E}[Z_t^2]\mathrm{E}[Z_{t+h}^2] \\ =& \sigma^4 \left\{\mathrm{E}[Y_t^2 Y_{t+h}^2] - \mathrm{E}[Y_t^2]\mathrm{E}[Y_{t+h}^2]\right\} = \sigma^4 \mathrm{Cov}(Y_t^2, Y_{t+h}^2).\end{aligned}$$

Hence, whenever Y_t is a dependent sequence, then $\mathrm{Cov}(Y_t^2, Y_{t+h}^2) \neq 0$ at least for some $h \neq 0$, so that $\gamma_{X^2}(t,t+h) \neq 0$.

Q4. (2 points) Let $\{X_t\}$ be a stationary sequence with the covariance function $\gamma_X(h)$. Let $\{Y_t\}$ be a stationary sequence with the covariance function $\gamma_Y(h)$. Assume also that the sequences $\{X_t\}$ and $\{Y_t\}$ are independent from each other. Find the covariance function of $X_t + Y_t$.

Solution to Q4:

Assume for simplicity that $\mathrm{E}[X_t] = \mathrm{E}[Y_t] = 0$.

$$\begin{aligned}&\gamma_{X+Y}(t,t+h) = \mathrm{Cov}(X_t + Y_t, X_{t+h} + Y_{t+h}) = \mathrm{E}[(X_t + Y_t)(X_{t+h} + Y_{t+h})] \\ &= \mathrm{E}[X_t X_{t+h}] + \mathrm{E}[X_t Y_{t+h}] + \mathrm{E}[X_{t+h} Y_t] + \mathrm{E}[X_{t+h} Y_{t+h}] = \gamma_X(h) + 0 + 0 + \gamma_Y(h)\ .\end{aligned}$$

Marking scheme for Q4:

Marking: 1 point for a partial answer, 2 points for full answer.

Q5. (2 points) Let $\{Z_t\}$ be independent random variables with mean 0 and variance σ^2. Determine if the following processes are stationary and causal.

(a) $X_t - 0.2X_{t-1} + 0.48X_{t-2} = Z_t$.

(b) $X_t + 1.6X_{t-1} = Z_t - 0.42Z_{t-1} + 0.04Z_{t-2}$.

Solution to Q5:

(a) The autoregressive polynomial is $\phi(z) = 1 - 0.2z + 0.48z^2$. The roots are $(0.2 - \sqrt{-1.88})/0.96 = 0.208 - 1.428i$ and $(0.2 + \sqrt{-1.88})/ = 0.208 + 1.428i$. Modulus of both roots are larger than one, so that there is a stationary and causal solution.

Marking: 1 point for looking only at real solutions, 1 point for correct answer in the complex domain. Total: 2 points.

(b) The autoregressive polynomial is $\phi(z) = 1 - 1.6z$. The root is $1/1.6$. Hence, the sequence is stationary but not causal.

Marking: this part will not be marked.