$1+x+x^2+\cdots = \frac{1}{1-x}$

$1+1/2+1/2^2+\cdots = 2$

$1+1/2^2+1/3^2+\cdots+1/n^2 = \pi^2/6$

$S = 1-1+1-1\cdots = 1/2$

$S = 1-(1-1+1\cdots) = 1-S$

$2S = 1 \Rightarrow S = 1/2$

III.1. A <u>sequence</u> is list of (real) numbers $\{s_1, s_2, s_3 \dots\} = \{s_n\}_{n=1}^{\infty}$

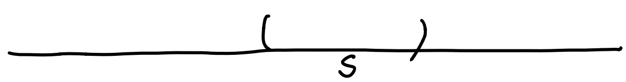
Example: $\{1, 1/2, 1/3, 1/4, \dots, 1/n\} = \{1/n\}_{n=1}^{\infty}$

$\{a, a+d, a+2d, a+3d, \dots\}$ Arithmetic progression

$\{1, r, r^2, r^3, \dots\}$ metric progression

Def. A sequence $\{s_n\}_{n=1}^{\infty}$ <u>convergence</u> if there exists a number $s \in \mathbb{R}$ such that

$$\forall_{\varepsilon>0}\ \exists_{N\geq 1}\ \forall_{n\geq N} \quad |s_n - s| < \varepsilon$$



The number s, if exists is called the limit of $\{s_n\}_{n=1}^{\infty}$ and we want $\lim_{n\to\infty} s_n = s$ or $s_n \to s$. If such an $s \in \mathbb{R}$ does not exist, we say that $\{s_n\}_{n=1}^{\infty}$ <u>divergence</u>.

Example:

Take $s_n = \frac{n}{n+1}$, $n = 1, 2, 3, \dots$

$\{s_n\}_{n=1}^{\infty} = \{1/2, 2/3, 3/4, \dots\}$

$\lim_{n\to\infty} S_n = 1$

$\forall_{\varepsilon>0}\ \exists_{N\geq 1}\ \forall_{n\geq N} \quad |s_n - 1| < \varepsilon$

$|s_n - 1| = \left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}$

we want $|s_n - 1| \leq \varepsilon$, i.e. $\frac{1}{n+1} < \varepsilon$

$\Rightarrow \frac{1}{\varepsilon} < n+1 \Rightarrow \frac{1}{\varepsilon} - 1 < n$

Thus we can set $N = \frac{1}{\varepsilon} - 1$

$N = \max\{\frac{1}{\varepsilon} - 1, 1\}$ $\qquad N = \frac{1}{\varepsilon} > \frac{1}{\varepsilon} - 1$

Example. $S_n = 1 + 1/2 + 1/2^2 + \cdots + 1/2^n \quad n \geq 1$

$$1 + 1/2 + 1/2^2 + \cdots = \lim_{n\to\infty} S_n$$

Prove that $\lim_{\infty} S_n = 2$

$$\forall_{\varepsilon>0}\ \exists_{N\geq 1}\ \forall_{n\geq N}\quad |S_n - 2| < \varepsilon$$

$$S_n = 1 + q + q^2 + \cdots + q^n, \quad q = 1/2$$
$$qS_n = q + q^2 + q^3 + \cdots + q^n + q^{n+1}$$
$$qS_n - S_n = q^{n+1} - 1$$
$$S_n = \frac{q^{n+1} - 1}{q - 1} = \frac{(1/2)^{n+1} - 1}{(1/2) - 1} = -2((1/2)^{n+1} - 1)$$
$$|S_n - 2| = |(2 - \tfrac{1}{2^n}) - 2|$$
$$= |-1/2^n| = 1/2^n$$

$$|S_n - 2| < \varepsilon \Rightarrow 1/2^n < \varepsilon \Rightarrow 1/\varepsilon < 2^n$$
$$\Rightarrow \log_2(1/\varepsilon) < n$$

Thus we can set $N = \max\{1, \log_2(1/\varepsilon)\}$

Example. Set $S_n = \frac{2n}{n+1}$. for $n = 1, 2, 3 \ldots$

show that $\lim_{n\to\infty} S_n = 2$

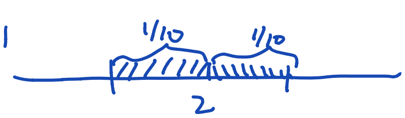
4: $\forall_{\varepsilon>0}\ \exists_{N\geq 1}\ \forall_{n\geq N}\quad |S_n - 2| < \varepsilon.$

$$|S_n - 2| = \left|\frac{2n}{n+1} - 2\right| = \left|\frac{2n - (2n+2)}{n+1}\right| = 2/n+1$$
$$|S_n - 2| < \varepsilon \Rightarrow \frac{2}{n+1} < \varepsilon \Rightarrow \frac{2}{\varepsilon} < n+1 \Rightarrow \frac{2}{\varepsilon} - 1 < n$$

Thus we set $N = \max\{2/\varepsilon - 1, 1\}$ will be the solution

$$\varepsilon = 1/10 \Rightarrow N > 2/(1/10) - 1 = 20 - 1 = 19$$
$$\Rightarrow N \geq 20$$

In particular, for $n \geq 10$, we have $|S_n - 2| < 0.1$



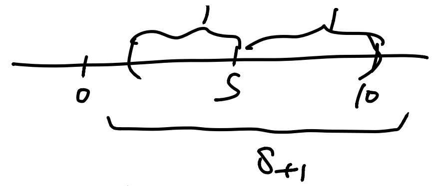
§1.3

Theorem: Let $\{S_n\}_{n=1}^{\infty}$ be a convergent sequences. Then $\{S_n\}_{n=1}^{\infty}$ is <u>bounded</u>

i.e. $\exists_{B>0}\ \forall_{n\geq 1},\ |S_n| \leq B$

proof: Assume that $\lim_{n\to\infty} S_n = s$ for some $s \in \mathbb{R}$

Thus $\forall \varepsilon > 0\ \exists N \geq 1\ \forall n \geq N\quad |S_n - s| < \varepsilon$



Set $\varepsilon = 1$. Then there exist $N_1 \geq 1$ such that $\forall n \geq N_1\quad |S_n - S| < 1$

$|S_n| - |S| < |S_n - S| < 1 \implies |S_n| < |S| + 1$

triangle inequality $|a| - |b| < |a - b|$

Set $B = \max\{|S|+1, |S_1|, \cdots |S_{N_1 - 1}|\}$

clearly $|S_n| \leq B$ for all $n \in \mathbb{N}$

Example $\{1, -1, 1, -1, \ldots \ldots\} = \{(-1)^{n-1}\}_{n=1}^{\infty}$

is bounded but not convergent

Def. we say a sequence $\{S_n\}_{n=1}^{\infty}$ tends to $+\infty$

if $\forall m > 0\ \exists N \geq 1\ \forall n \geq N\quad S_n > m$

we say a sequence $\{S_n\}_{n=1}^{\infty}$ tends to $-\infty$

if $\forall m < 0\ \exists N \geq 1\ \forall n \geq N\quad S_n < m$

Example (i) $S_n = n\ (n = 1, 2, 3, \cdots)$

Then $\lim_{n\to\infty} S_n = +\infty$

Take any m, we need $S_n > m$

$\Updownarrow$

$n > m$

Then we can get $N = m$

(ii) Set $S_n = -n^2\ (n = 1, 2, 3, \cdots)$

Then $\lim_{n\to\infty} S_n = -\infty$

Take any $m < 0$, $S_n < m$

$\Updownarrow$

$-n^2 < m \implies n^2 > -m\ (>0) \overset{(n>0)}{\Longleftrightarrow} n > \sqrt{-m}$

Hence it suffices to set $N = \max\{1, \sqrt{-m}\}$