COMP 233/2

Probability and Statistics for Computer Science

Week 5

Bernoulli and Binomial random variable
Poisson random variable
Exponential random variable
Uniform random variable

Reading: Chap 5

Food for Thought – Example 3.1

- A set of k coupons is collected. Each coupon is one of n types. Each coupon is independently a type j coupon with probability p_j such that $\sum_{j=1}^{n} p_j = 1$. Find the probability that the set contains either a type 1 or type 2 coupon.

Solution

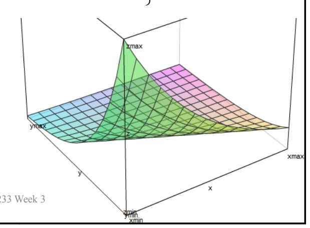
- C_i : ith coupon in the set is of type 1 or 2
- $C = \bigcup_{i}^{k} C_i$
- Probability C_i is neither type 1 nor 2 is (1-p1-p2) = $P(C_i^c)$
- Probability that none of the coupons in the set is of type 1 or 2, by independence is: $\prod_{i}^{k} P(C_i^c)$ = (1-p1-p2)^k
- Probability that the set has at least one coupon of type 1 or 2 is therefore 1 - (1-p1-p2)^k

Food for Thought – Example 3.2

- Suppose the joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x, 0 < y \\ 0 & \text{otherwise} \end{cases}$$

- Compute
  (a) P{X>1, Y<1}
  (b) P{X<Y}



Solution for Food for Thought – Example 3.2

(a) $$P\{X>1, Y<1\} = \int_0^1\int_1^\infty 2e^{-x}e^{-2y}\,dxdy$$

$$= \int_0^1 2e^{-2y}\left(-e^{-x}\Big|_1^\infty\right)dy$$

$$= e^{-1}\int_0^1 2e^{-2y}\,dy$$

$$= e^{-1}(1-e^{-2})$$

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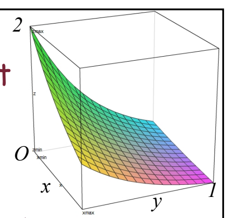
Food for Thought



(b) $$P\{X<Y\} = \int_0^\infty\int_0^y 2e^{-x}e^{-2y}\,dxdy$$

$$= \int_0^\infty 2e^{-2y}\left(-e^{-x}\Big|_0^y\right)dy = \int_0^\infty 2e^{-2y}\left(-e^{-y}+1\right)dy$$

$$= \int_0^\infty 2e^{-2y}\,dy - \int_0^\infty 2e^{-3y}\,dy$$

$$= 1-\frac{2}{3} = \frac{1}{3}$$

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Food for Thought – 3.3

- Decide if the following is the joint density function of independent RVs.

$$f(x,y) = \frac{1}{4\pi}e^{-\left(\frac{x^2}{2}+\frac{y^2}{8}\right)}, \qquad x,y \in \mathbb{R}$$

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Solution Food for Thought 3.3

- We observe that

$$f(x,y) = \frac{1}{4\pi}e^{-\left(\frac{x^2}{2}+\frac{y^2}{8}\right)}$$

$$= \left(\frac{1}{\sqrt{2\pi}}e^{-\left(\frac{x^2}{2}\right)}\right)\left(\frac{1}{\sqrt{8\pi}}e^{-\left(\frac{y^2}{8}\right)}\right)$$

$$= f_X(x)f_Y(y)$$

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Solution

- Can also confirm that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{4\pi}e^{-\left(\frac{x^2}{2}+\frac{y^2}{8}\right)}dxdy = 1$$

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\left(\frac{x^2}{2}\right)}dx = 1$$

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{8\pi}}e^{-\left(\frac{y^2}{8}\right)}dy = 1$$

So $f(x,y)$ is the product of two PDFs of two independent RVs.

Food for Thought – 4.1

- Consider the following joint PMF:

i \ j	j=0	1	2	3	P{X=i}
i=0	0.1	0.1	0.05	0.05	0.3
1	0.1	0.1	0.05	0.05	0.3
2	0.1	0.05	0.025	0.025	0.2
3	0.1	0.05	0.025	0.025	0.2
P{Y=j}	0.4	0.3	0.15	0.15	1

- Determine $Var(X), Var(Y), Cov(X,Y)$, and $Var(X+Y)$.

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Food for Thought – 4.2

- Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50.

  (a) What can be said about the probability that this week's production will *exceed* 75?

  (b) If the variance of a week's production is known to equal 25, then what can be said about the probability that this week's production will be *between* 40 and 60?

Solution

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

- Let X be the number of items that will be produced in a week:

  (a) By Markov's inequality (substituting 75 for a),

$$P\{X \geq 75\} \leq \frac{E[X]}{75} = \frac{50}{75} = \frac{2}{3}$$

Solution $P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$

(b) By Chebyshev's inequality

$$P\{|X-50| \geq 10\} \leq \frac{\sigma^2}{10^2} = \frac{1}{4}.$$

Then

$$P\{|X-50| < 10\} \geq 1 - \frac{1}{4} = \frac{3}{4},$$

so the probability that this week's production will be between 40 and 60 is at least .75.

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The Weak Law of Large Numbers

- Let $X_1, X_2, \ldots$, be a sequence of independent and identically distributed random variables, each having mean $E[X_i] = \mu$. Then, for any $\varepsilon > 0$,

$$P\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \varepsilon\right\} \to 0 \text{ as } n \to \infty$$

- In other words, for any positive number ε, no matter how small, the probability that the proportion of the first n trials in which an event occurs differs from the mean by more than ε goes to 0 as n increases.

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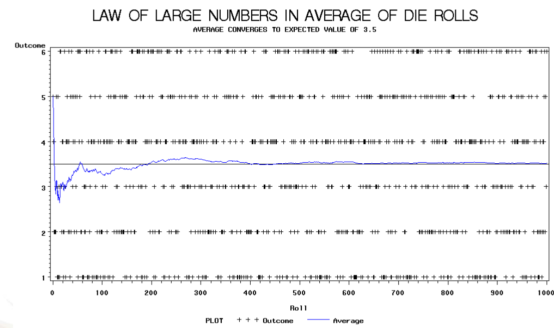
Illustration



Source: http://en.wikipedia.org/wiki/Law_of_large_numbers

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Food for Thought – 4.3

- We can use the Chebyshev inequality to prove the Weak Law of Large Numbers.
- That is use the Chebyshev inequality,

$$P\{|X-\mu| \geq k\} \leq \frac{Var(X)}{k^2}$$

And prove:

$$P\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \varepsilon\right\} \to 0 \text{ as } n \to \infty$$

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Solution

$$P\left\{\left|\frac{X_1+\cdots+X_n}{n}-\mu\right|>\varepsilon\right\}=P\left\{\left|\bar{X}-\mu\right|>\varepsilon\right\}\leq\frac{Var(\bar{X})}{\varepsilon^2}$$

Note that

$$Var(\bar{X})=Var\left(\frac{X_1+\cdots+X_n}{n}\right)$$

$$=Var\left(\frac{X_1}{n}\right)+\cdots+Var\left(\frac{X_n}{n}\right)=\frac{\sigma^2}{n^2}+\cdots+\frac{\sigma^2}{n^2}=\frac{\sigma^2}{n}$$

So

$$P\left\{\left|\frac{X_1+\cdots+X_n}{n}-\mu\right|>\varepsilon\right\}\leq\frac{\sigma^2}{n\varepsilon^2}\to 0 \text{ as } n\to\infty$$

Special Random Variables

- Certain types of random variables occur over and over again in applications. This week and next, we will study a variety of them.

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Bernoulli Experiments

1654 –1705

- A ***Bernoulli*** trial is an experiment with two outcomes: S(uccess) and F(ailure).

  $P\{S\} = p$ and $P\{F\} = q$ $(= 1-p)$.
- Example: Tossing a coin.
  - S and F are contextually defined.
  - For a fair coin, $p = q = 0.5$.
- Example: Receiving a bit; error probability = 0.1.
  - So, $p = 0.1$, so $q = 0.9$.

Bernoulli Random Variables

- A ***Bernoulli*** random variable, Y, is defined by:
  - $Y = 1$, if the outcome is Success,
  - $Y = 0$, if the outcome is Failure.

Trial #	1	2	3	4	5	6	7
Value of Bernoulli RV	1	1	0	1	0	0	1

Bernoulli Random Variables

- The ***Bernoulli*** probability mass function of Y is given by the following table:

Y	1	0
$P\{Y\}$	p	$1 - p$

Expectation and Variance of Bernoulli Random Variable

- A direct computation yields

$$E[Y] = 1 \cdot p + 0 \cdot (1-p)$$
$$= p,$$
$$Var(Y) = \left[1^2 \cdot p + 0^2 \cdot (1-p)\right] - p^2$$
$$= p(1-p)$$
$$= pq.$$

The Binomial Mass Function

- A ***binomial*** random variable X, with parameters (n, p), represents the number of successes in *n* independent *Bernoulli trials* (with same p).
- Note that

$$P\{X = \mathrm{k}\} = \binom{n}{k} p^{\mathrm{k}} (1-p)^{n-\mathrm{k}},$$

$$\mathrm{k} = 0, 1, 2, \ldots, n.$$

Correctness

- The validity of the Binomial PMF may be verified by first noting that the probability of any particular sequence of the n outcomes containing k successes and n – k failures is, by the assumed independence of trials, $p^k(1 - p)^{n-k}$.
- The PMF then follows since there are ${}^nC_k = \binom{n}{k}$ different sequences of the n outcomes leading to k successes and n – k failures—which can perhaps most easily be seen by noting that there are ${}^nC_k$ different selections of the k trials that result in successes.

Correctness, cont.

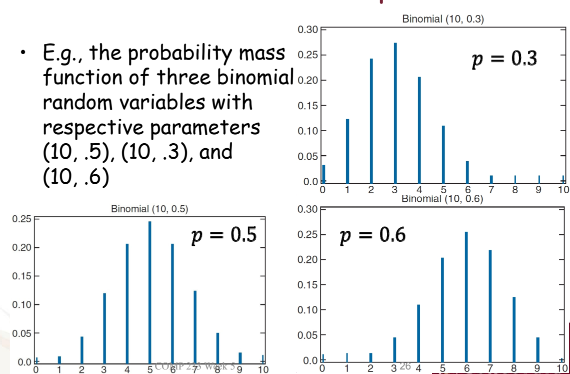
- For instance, if n = 5, k = 2, then there are $\binom{5}{2}$ choices of the two trials that are to result in successes — namely, any of the outcomes

  (s, s, f, f, f) (f, s, s, f, f) (f, f, s, s, f)
  (s, f, s, f, f) (f, s, f, s, f) (f, f, s, f, s)
  (s, f, f, s, f) (f, s, f, f, s) (f, f, f, s, s)
  (s, f, f, f, s)

  where the outcome (f, s, f, s, f) means, for instance, that the two successes appeared on trials 2 and 4.

Binomial PMF Graphs

- E.g., the probability mass function of three binomial random variables with respective parameters (10, .5), (10, .3), and (10, .6)



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Binomial Distribution

- A ***binomial*** random variable, X, represents the number of Successes in *n* independent *Bernoulli trials*.
- Hence,

$$X = Y_1 + Y_2 + \ldots + Y_n,$$

where Y_i is the Bernoulli RV.

Expectation and Variance of the Binomial RV

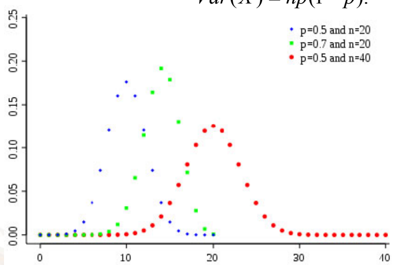
- For a *binomial* RV with parameters *n* and *p* we have

$$E[X] = np,$$
$$Var(X) = np(1-p) = npq.$$

  - Try working out these results directly!
  - Think of how *Bernoulli* and *binomial* RV are related.

Binomial Distribution

- Binomial PMF: $E[X] = np,$
  $Var(X) = np(1-p).$



Example

- It is known that disks produced by a certain company will be defective with probability .01 independently of each other. The company sells the disks in packages of 10 and offers a money-back guarantee that *at most 1* of the 10 disks is defective.

a) What proportion of packages is returned?

b) If someone buys three packages, what is the probability that exactly one of them will be returned?

Solution

a) If X is the number of defective disks in a package, then assuming that customers always take advantage of the guarantee, it follows that X is a binomial random variable with parameters (10, .01). Hence the probability that a package will have to be replaced is

$$P\{X>1\} = 1 - P\{X=0\} - P\{X=1\}$$

$$= 1 - \binom{10}{0}(.01)^0(.99)^{10} - \binom{10}{1}(.01)^1(.99)^9$$

$$\approx .005$$

Solution, cont.

- Because, from a), each package will, independently, have to be replaced with probability .005, it follows from the weak law of large numbers that in the long run .5 percent of the packages will have to be replaced.
- It follows then that the number of packages that the person will have to return is a binomial random variable with parameters n = 3 and p = .005.
- Therefore, the probability that exactly one of the three packages will be returned is

$$\binom{n}{1}(p)^1(1-p)^{n-1} = \binom{3}{1}(.005)(.995)^2 = .015$$

Example

- The color of one's eyes is determined by a single pair of genes, with the gene for brown eyes being dominant over the one for blue eyes. This means that an individual having two blue-eyed genes will have blue eyes, while one having either two brown-eyed genes or one brown-eyed and one blue-eyed gene will have brown eyes. When two people have children, the resulting offspring receives one randomly chosen gene from each of its parents' gene pair.
- If the eldest child of a pair of brown-eyed parents has blue eyes, what is the probability that exactly two of the four other children (none of whom is a twin) of this couple also have blue eyes?

Solution

- To begin, note that since the eldest child has blue eyes, it follows that both parents must have one blue-eyed and one brown-eyed gene. The probability that an offspring of this couple will have blue eyes is equal to the probability that it receives the blue-eyed gene from both parents, which is (1/2)(1/2) = 1/4.
- Hence, because each of the other four (n) children will have blue eyes with probability 1/4, it follows that the probability that exactly two (k) of them have this eye color is

$$\binom{4}{2}(1/4)^2(3/4)^2 = 27/128$$

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The Geometric Distribution [not in textbook]

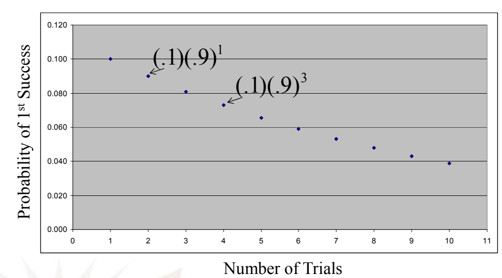
- Suppose an experiment consists of repeating independent *Bernoulli trials,* until the first Success.
- A ***geometric*** random variable, X, represents the number of trials until the 1st success.
- Note that

$$P\{X = k\} = p(1-p)^{k-1},\ k \geq 1.$$

$$F(k) = P\{X \leq k\} = 1-(1-p)^k.$$

- *How do we derive this?*

PMF Graph (Geometric p=0.1)



The Geometric Distribution [not in textbook]

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- Note that

$$P\{X = k\} = p(1-p)^{k-1},\ k \geq 1.$$

$$F(k) = P\{X \leq k\} = 1-(1-p)^k.$$

- *How do we derive this?*

$$F(k) = P\{X \leq k\}$$
$$= 1 - P\{X > k\}$$

where

$$P\{X>k\} = \sum_{i=k+1}^{\infty} p(1-p)^{i-1}$$
$$= p\left[(1-p)^k + (1-p)^{k+1} + (1-p)^{k+2} + \dots\right]$$
$$= p(1-p)^k\left[(1-p)^0 + (1-p)^1 + (1-p)^2 + \dots\right]$$
$$= p(1-p)^k\left(\frac{1}{1-(1-p)}\right)$$
$$= (1-p)^k$$
$$\therefore F(k) = 1-(1-p)^k$$

Expectation and Variance of the Geometric RV

- For a geometric RV with parameter p we have

$$E[X] = \frac{1}{p},$$

$$Var(X) = \frac{1-p}{p^2}.$$

E[X] = Sum i=1 to k (i.p.(1-p)^i-1)
= 1.p + 2.p.(1-p) + 3.p.(1-p)^2 + …
(1-p)E[X] = p(1-p) + 2p(1-p)^2 +3p(1-p)^3 + …
E[X] – (1-p)E[X] = p.E[X]
= p((1-p)^0 + (1-p)^1 + (1-p)^2 + …)
= p(1/p) = 1
p.E[X] = 1 so, E[X] = 1/p

Food for Thought 5.1

- A representative from the Canadian Football League's Marketing Division randomly selects people on an arbitrary street in Montreal until he finds a person who attended the last home football game. Let p, the probability that a person has attended the last home game, equal 0.20. And, let X denote the number of people he selects until he finds his *first* success.

a) What is the probability that the marketing representative must select 4 people before he finds one who attended the last home football game?

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Food For Thought, cont.

b) What is the probability that the marketing representative must select *more than* 6 people before he finds one who attended the last home football game?

c) How many people should we expect (that is, what is the average number) the marketing representative needs to select before he finds one who attended the last home football game?

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Example

- Suppose the first *k* trials were Failures.
  - Find the probability that (*k+1*)-th trial will be a Success.
- To find is $P\{X = k+1 \mid X > k\}$.
- The definition of *conditional* probability and direct computation imply that

$$P\{X = k+1 \mid X > k\} = \frac{p(1-p)^k}{(1-p)^k} = p = P\{X = 1\}.$$

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The Negative Binomial Distribution [not in text]

- Suppose an experiment consists of repeating independent *Bernoulli trials,* until there are r successes.
- A ***negative binomial*** random variable, X, represents the number of trials until the rth success.
- Note that

$$P\{X = k\} = \binom{k-1}{r-1} p^r (1-p)^{k-r},\ k \geq 1.$$

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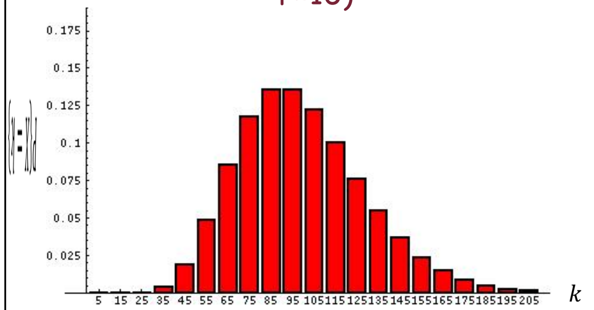
PMF (Negative Binomial, p=0.1, r=10)



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Graph from: www.zoology.ubc.ca/~bio301/Bio301/Lectures/Lecture23/Overheads.html

Expectation and Variance of the Negative Binomial RV

- For a negative binomial RV with parameter *p* we have

$$E[X] = r\left(\frac{1}{p}\right),$$

$$Var(X) = r\left(\frac{1-p}{p^2}\right).$$

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Example

- Example: If a predator must capture 10 prey before it can grow large enough to reproduce, what would the mean age of onset of reproduction be if the probability of capturing a prey on any given day is 0.1? Mean = 10/0.1 = 100.
- Notice that again the variance in this case is quite high (~1000) and that the distribution looks quite skewed (=not symmetric). Some predators will reach reproductive age much sooner and some much later than the average

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Food for Thought 5.2

- *An oil company conducts a geological study in Alberta that indicates that an exploratory oil well should have a 20% chance of striking oil.*
- *What is the probability that the third strike comes on the seventh well drilled?*
- *What is the mean number of wells that must be drilled if the oil company wants to set up three producing wells?*

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Motivating Example

- When a bit is transmitted, it is received in error with probability 0.0001.
- Assume that 25000 bits are transmitted independently.
  - What is the Mass Function of the number of bits received in error?
  - What is the probability that not more than 5 bits are received in error?

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Binomial Mass Function with large *n* and small *p*

- The number of erroneous bits is *binomially distributed* with $n=25000$ and $p=0.0001$.
- We have $E[X] = np = 2.5$ and $Var(X) = npq = 2.49975$.
- A direct computation of associated probabilities requires ${}^{25000}C_k$ and 0.9999^k.
- Is there a more efficient way to compute such probabilities?
- We shall take a look at similar examples for insight.

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Binomial Mass Function with large *n* and small *p*

- The number of erroneous bits is binomially distributed with n=25000 and p=0.0001
- We need an efficient way to compute such probabilities.
- So next topic is

POISSON Mass Function

AND

RELATED RANDOM VARIABLES

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Poisson Random Variable

1781 –1840

Example 1

- On *average* 3 messages arrive at a server per minute. Assume that messages arrive independently.

Example 1

- On average 3 messages arrive at a server per minute. Assume that messages arrive independently.
  - For a randomly selected 1-minute interval, find the probability that exactly 2 messages arrive.
  - Can we try solving the problem by the binomial Mass Function?

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Solution

- We divide the 1-minute interval into sub-intervals, with the length of, say, 1ms each, i.e., into 60000 sub-intervals of equal length.
- The probability of 2 or more messages arriving within 1 ms is negligible.
- Hence, each sub-interval contains either 1 message or 0 messages.
- For each sub-interval, the probability of having a message equals 3/60000 = 0.00005.

Solution

- Hence, binomial Mass Function can be used:

$$^{60000}C_2 \; 0.00005^2 \; 0.99995^{59998} = P\{X=2\}.$$

- However, this is an approximate value.
- *Can we compute the requested probability in such a manner that the answer is independent of partitioning?*

1st	2nd	3rd	…	…	…	59999	60000
0	1	0	…	…	…	0	1

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Observation

- Assume that we divide the 1-minute interval into *n* sub-intervals of equal length, with *large n*.
- The probability of 2 or more messages arriving within *1/n* minutes is negligible.
- Hence, each sub-interval contains either 1 message or 0 messages.
- For each sub-interval, the probability of having a message equals p = *3/n*.

Observation

Then

$$P\{X=2\} = \binom{n}{2}(\frac{3}{n})^2(1-\frac{3}{n})^{n-2}$$

$$= \frac{n(n-1)}{2!}\cdot(\frac{3}{n})^2\cdot(1-\frac{3}{n})^n\cdot(1-\frac{3}{n})^{-2}$$

$$= \frac{n(n-1)}{n^2}\cdot\frac{3^2}{2!}\cdot(1-\frac{3}{n})^n\cdot(1-\frac{3}{n})^{-2}$$

- What happens if we keep increasing *n*, i.e., *pass to the limit* as *n* approaches infinity?

Solution

- We get

$$P[X=2] \to e^{-3}\frac{3^2}{2!} \text{ as } n\to\infty.$$

- The expression for *P[X=2]* is now independent of *n*.
- In general,

$$P\{X=\text{k}\} \to e^{-\lambda}\frac{\lambda^\text{k}}{\text{k}!} \text{ as } n\to\infty.$$

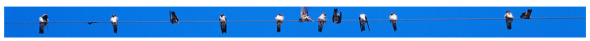
General Case

- In general, let counts occur at random throughout an interval on the real line. If the interval can be divided into sub-intervals such that
  - The probability of more than 1 count in one sub-interval is 0;
  - The probability of 1 count in a sub-interval is the same for all sub-intervals, and proportional to the length of the sub-interval;
  - The count in each sub-interval is independent of counts in other sub-intervals.
- Then the number of counts N_λ in the interval is said to have ***Poisson*** Mass Function with parameter ***λ>0***.

$$P\{N_\lambda = k\} = e^{-\lambda}\frac{\lambda^k}{k!}, k=0,1,2...$$

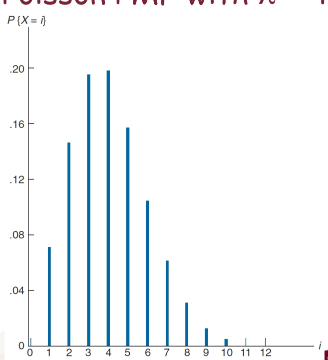
Examples of Poisson RV

- For example, the number of
  - Calls to a telephone exchange during an hour;
  - Transactions on a network per minute;
  - Contamination particles in semiconductor production;
  - Flaws in a band of magnetic tape;
  - Raisins in a slice of English cake;
  - Birds on 100 meter of the wire;



can be modeled using *Poisson* RV.

Poisson PMF with λ = 4.



Approximating Binomial Probabilities

- In view of the limiting argument (where n goes to infinity), the *Poisson* Mass Function provides an *excellent approximation to binomial probabilities* in the case of large *n* and small *p*.
- We simply set $\lambda = np$.

Example

- Suppose the probability that an item produced by a certain machine will be defective is 0.1. Find the probability that a sample of 10 items will contain at most one defective item. Assume that the quality of successive items is independent.
- **Solution.** The desired probability is

$$\binom{10}{0}(.1)^0(.9)^{10} + \binom{10}{1}(.1)^1(.9)^9 = .7361,$$

whereas the Poisson approximation yields the value

$$e^{-1}\frac{1^0}{0!} + e^{-1}\frac{1^1}{1!} \approx .7358$$

Observation

- Note that

$$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!} \qquad \text{(series expansion)}$$

- Hence,

$$\sum_{k=0}^{\infty} P\{N_\lambda = k\} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1$$

Reminder: Expectation and Variance of Binomial RV

- For a *binomial* RV with parameters *n* and *p* we have

$$E[X] = np$$

$$Var(X) = np(1-p)$$

- Try guessing the expected value and the variance of Poisson Mass Function with $\lambda>0$?

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Expectation and Variance of Poisson Mass Function

- For a *Poisson* RV with parameter $\lambda>0$ we have

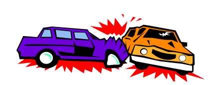
$$E[N_\lambda] = \lambda,$$

$$Var(N_\lambda) = \lambda.$$

  - Try working these results out directly!
  - Hint: use the power series expansion for an exponential function.

Example

- Suppose that the *average* number of accidents occurring weekly on a particular stretch of a highway equals 3. Calculate the probability that there is at most two accidents this week.



- Let N_λ denote the number of accidents occurring on the stretch of highway in question during this week.

Solution

- Because it is reasonable to suppose that there are a large number of cars passing along that stretch, each having a small probability of being involved in an accident, the number of such accidents should be approximately Poisson distributed. Note that $\lambda = E[N_\lambda] = 3$.
- Then, $P\{N_\lambda = 0\} + P\{N_\lambda = 1\} + P\{N_\lambda = 2\}$

$$= e^{-\lambda}\left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!}\right) = e^{-3}\left(\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!}\right)$$

$$\approx (0.04979)(1 + 3 + 4.5) \approx .4232$$

Exponential Random Variable

2/3/2020

Example 1 (re-visited)

- On average 3 messages arrive at a server per minute.
- Assume that messages arrive independently.
  - Find the probability that at most 30 seconds lie between 2 consecutive messages.

Comments

- The sample space has changed! We are now interested in the TIME between consecutive messages.
- Needless to say, the time values are also random (just as the counts of messages are), however the values fill an INTERVAL on the number line.
- The sample space and the respective random variable are therefore CONTINUOUS.
- It is also clear that a continuous Density Function of this type naturally arises from a Poisson Mass Function.

Solution

- So let us try using Poisson probabilities to work out the answer.
- Let L stand for the time between consecutive messages.
- We observe first that $P\{L \leq 0.5\} + P\{L > 0.5\} = 1$.

Solution

- 3 messages per minute → 1.5(=3/2) messages per 30 seconds, *on average*.
- That is the number of messages per 30 seconds is Poisson distributed with λ=1.5.

$$P\{N_{1.5} = \mathrm{k}\} = e^{-1.5}\frac{1.5^{\mathrm{k}}}{\mathrm{k}!}, \ \mathrm{k} = 0,1,2...$$

2/3/2020

Solution

If the time between 2 messages exceeds 30 seconds,
then NO MESSAGE ARRIVED WITHIN 30 seconds.

- So the respective Poisson RV assumes a value of 0.
- In other words, $P\{L > 0.5\} = P\{N_{1.5} = 0\} = e^{-1.5}$.
- And $P\{L \leq 0.5\} = 1 - P\{L > 0.5\} = 1 - e^{-1.5}$
  $\approx 1 - 0.223 = 0.777$.

General Case

- λ messages per minute → λt messages within t minutes on average.
- That is the number of messages within t minutes is Poisson distributed with parameter λt.
- If the time between 2 messages exceeds t minutes, then NO MESSAGE ARRIVED WITHIN within t minutes.
- Hence, $P\{L > t\} = P\{N_{\lambda t} = 0\} = e^{-\lambda t}$.
- And $P\{L \leq t\} = 1 - P\{L > t\} = 1 - e^{-\lambda t}$.

The Exponential Density Function

- Suppose counts in an interval on the number line follow Poisson Density Function with a mean of $\lambda>0$.
- Then the *lengths of intervals* between successive counts are said to be ***exponentially*** distributed with $\lambda>0$ and

$$P\{L \le t\} = 1 - P\{L > t\} = 1 - e^{-\lambda t},\ t \ge 0.$$

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Exponential Density Function, review

- Recall the ***PDF*** of the *exponential* RV with $\lambda>0$,

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \lambda e^{-\lambda t} & \text{if } t \ge 0 \end{cases}$$

- A direct computation yields the CDF (from week 3)

$$F(t) = P\{L \le t\} = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-\lambda t} & \text{if } t \ge 0 \end{cases}$$

- As shown before, $E[L] = 1/\lambda$, $Var(L) = 1/\lambda^2$.

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2/3/2020

Example 1 (re-visited)

- *On average 3 messages arrive at a server per minute.*
- *Assume that messages arrive independently.*
  - If no message arrived within 24 seconds, find the probability that no message will arrive in the next 15 seconds.

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Solution

- To find is $P\{L>0.4+0.25|L>0,4\}$!
  - *24 sec. = 0.4 min., 15 sec. = 0.25 min.*
  - Requested is a *conditional* probability.
  - It is given by

$$\frac{P\{(L>0.4+0.25)\ and\ (L>0.4)\}}{P\{L>0.4\}} = \frac{P\{L>0.65\}}{P\{L>0.4\}}$$

$$= \frac{e^{-3\cdot 0.65}}{e^{-3\cdot 0.4}} = e^{-3\cdot 0.25} = P\{L>0.25\}$$

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General Case

- In general, for an exponential Density Function with λ>0, we have

$$P\{L > t + s \mid L > t\} = P\{L > s\}.$$

- Hence, exponential RV possess the memoryless property.
- One can show that a continuous Density Function has the memoryless property only if it is exponential !

Example

- Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles.
- If a person desires to take a 5,000-mile trip, what is the probability that she will be able to complete her trip without having to replace her car battery?

Solution

- It follows, by the memoryless property of the exponential distribution, that the remaining lifetime (in thousands of miles) of the battery is exponential with parameter λ = 1/10 (since $E[L]$ = 1/λ).
- So the desired probability is

$$P\{\text{remaining lifetime} > 5\} = 1 - F(5)$$
$$= e^{-\lambda 5}$$
$$= e^{-1/2} \approx .604$$

Memory-less Property of Geometric Mass Function

- Note that for a geometric mass function, with p>0, we have

$$P\{X = k+1 \mid X \geq k\} = P\{X = 1\}.$$

- That is, geometric RV possess the memoryless property, too.
- It appears, that if a discrete mass function has the memoryless property, then it is geometric.

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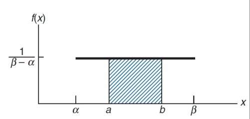
Uniform Random Variable

Uniform Density Function

- As seen previously, a continuous RV is said to be ***uniformly*** distributed ***on*** *[α, β]* if its PDF is

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \alpha \le x \le \beta, \\ 0, & \text{otherwise} \end{cases}$$

- Then $P\{a \le X \le b\} = \dfrac{b-a}{\beta - \alpha}$



- And $E[X] = \dfrac{\alpha + \beta}{2}, \quad Var(X) = \dfrac{(\beta - \alpha)^2}{12}$

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Food for Thought 5.3

- Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7:00, 7:15, 7:30, 7:45, and so on.
- If a passenger arrives at the stop at a time that is uniformly distributed between 7:00 and 7:30, find the probability that he waits
  a) less than 5 minutes for a bus,
  b) at least 12 minutes for a bus.

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Random Numbers

- The value of a uniform (0, 1) random variable is called a random number. Most computer systems have a built-in subroutine for generating (to a high level of approximation) sequences of "independent" random numbers.
- Random numbers are quite useful in probability and statistics because their use enables one to empirically estimate various probabilities and expectations.
- More in final week of the course.

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References/Resources Used

- Lecture Slides for MATH 401 of Dr. Oleksiy Us, Department of Mathematics, German University of Cairo. [PPT]

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