SOLUTIONS

MAT1330 B – Instructor: Elizabeth Maltais
Wednesday, October 3, 2017 : Test #1

Duration: 75 minutes

Family name: ________________________________

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For picking up your graded test:
Circle the DGD you will attend to pick up your test (whether you are registered or not). Tests take at least one week to grade; watch for announcements in class.

# :	DGD1	DGD2	DGD3	DGD4
Day :	Tuesday	Wednesday	Wednesday	Thursday
Start time:	10:00	10:00	1:00	11:30
Room :	MRT 250	SMD 224	STE J0106	FTX 133
TA :	Eric	Rabib	Rabib	Rabib

Please read the following instructions carefully.

- You have 75 minutes to complete this exam.
- This is a closed book exam. Except for Faculty approved calculators (models: Texas Instruments TI-30* and TI-34*, Casio FX-260* and Casio FX-300*), no notes, cell phones, smart watches or related devices of any kind are permitted. All such devices, including cell phones, **must be stored in your bag under your desk or at the front of the room for the duration of the exam**.
- Read each question carefully — you will save yourself time and grief later on.
- Questions 1 through 6, and 8, are multiple choice, worth a total of 7 points. **Record your answers to the multiple choice questions in the boxes provided.**
- Questions 7, 9, 10 and 11 are long answer, with number of points as indicated. **You must show your work, your work must be legible and well-justified, and you must record your answers in the spaces provided.**
- Where it is possible to check your work, do so.
- Good luck!

Marker's use only:

Question	Marks
1–6,8 (/7)	
7 (/2)	
9 (/2)	
10 (/5)	
11 (/4)	
Total (/20)	

SAMPLE TEST SOLUTIONS

1. (1 point) Find all real solutions of $|1-2x^2| = 7$.

[✓ A. ± 2]

B. $\pm\sqrt{2}$

C. No solutions exist.

D. ± 3

E. $\pm\sqrt{-3}$

F. 0

Your answer: A

$$|1-2x^2| = 7 \Rightarrow 1-2x^2 = 7 \quad \text{OR} \quad 1-2x^3 = -7$$

$$\Rightarrow \underbrace{-6 = 2x^2}_{\text{NO SOLUTIONS}}$$

$$8 = 2x^2$$

$$4 = x^2$$

$$x = \pm 2$$

2. (1 point) What is the domain of $f(x) = \dfrac{\ln(1-2x)}{\sqrt{x+1}}$?

A. all x

B. $x > 0$

[✓ C. $-1 < x < \frac{1}{2}$]

D. $x \geq \frac{1}{2}$

E. $x > -1$

F. $\frac{1}{2} < x \leq 1$

Your answer: C

we need $1-2x > 0$ and $x+1 > 0$

$\Rightarrow$ $1 > 2x$ $\qquad$ $x > -1$

$\frac{1}{2} > x$

$\therefore -1 < x < \frac{1}{2}$

3. (1 point) Which of the following is equal to $g(x) = \ln(e^x x^3)$ on its domain?

A. $3x\ln(x)$

B. $3x$

C. e^3

D. $1 + 3\ln(x)$

[✓ E. $x + 3\ln(x)$]

F. $\ln(3e)$

Your answer: E

$$\ln(e^x x^3) = \ln(e^x) + \ln(x^3)$$

$$= x + 3\ln(x)$$

4. (1 point) Which of the following is equal to $h(x) = \dfrac{\sqrt{9x^6+4}}{2x^3}$ for $x > 0$?

- [✓ A. $\sqrt{\frac{9}{4} + x^{-6}}$]
- B. $\frac{3}{2} + 2x^{-3}$
- C. $\sqrt{4.5 + x^{-3}}$
- D. $\frac{3}{\sqrt{2}} x^{3/2} + x^{-3/2}$
- E. $\sqrt{26x^{12} + 16x^6}$
- F. $3x^3 + 2x^{-3}$

Your answer: A

$$\frac{\sqrt{9x^6+4}}{2x^3} = \frac{\sqrt{9x^6+4}}{\sqrt{4x^6}} = \sqrt{\frac{9x^6+4}{4x^6}} = \sqrt{\frac{9x^6}{4x^6} + \frac{4}{4x^6}}$$

$$= \sqrt{\frac{9}{4} + x^{-6}}$$

5. (1 point) Does there exist a choice of a which makes the following function continuous at $x = 1$? If yes, which one? If not, why not?

$$f(x) = \begin{cases} \ln(2ax) & \text{if } x > 1 \\ \cos(\pi x) & \text{if } x \leq 1 \end{cases}$$

- $\ln(2ax)$ is continuous for all $x > 1$ and $a > 0$

$\cos(\pi x)$ is continuous for all $x \leq 1$

∴ the only issue for continuity would be at $x = 1$.

- A. yes, $a = e$
- [✓ B. yes, $a = \frac{1}{2}e^{-1}$]
- C. yes, $a = \frac{1}{2}$
- D. yes, $a = 0$
- E. no, $\lim_{x\to 1} f(x)$ exists, but is not equal to $f(1)$ for any choice of a so f can never be continuous at 1
- F. no, $\lim_{x\to 1} f(x)$ does not exist for any choice of a, so f can never be continuous at 1

Your answer: B

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \ln(2ax) = \ln(2a(1)) = \ln(2a)$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \cos(\pi x) = \cos(\pi \cdot 2) = \cos(2\pi) = -1$$

To be continuous at $x = 1$, we need $\lim_{x\to 1} f(x) = f(1) = -1$

∴ we need $\ln(2a) = -1 \Rightarrow 2a = e^{-1} \Rightarrow a = \frac{1}{2}e^{-1}$

6. (1 point) If $h(x) = \dfrac{1006 + 222x + 317x^2}{4024 + 554x + 634x^2}$ then

[✓ A. $\lim\limits_{x\to\infty} h(x) = \dfrac{1}{2}$]

B. $\lim\limits_{x\to\infty} h(x) = \dfrac{1}{4}$

C. $\lim\limits_{x\to\infty} h(x) = 1$

D. $\lim\limits_{x\to\infty} h(x) = 0$

E. $\lim\limits_{x\to\infty} h(x) = -\infty$

F. $\lim\limits_{x\to\infty} h(x) = \infty$

Your answer: A

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{1006 + 222x + 317x^2}{4024 + 554x + 634x^2} = \lim_{x\to 0} \frac{\cancel{x^2}\left(\frac{1006}{x^2} + \frac{222}{x} + 317\right)}{\cancel{x^2}\left(\frac{4024}{x^2} + \frac{554}{x} + 634\right)} = \frac{0+0+317}{0+0+634} = \frac{1}{2}$$

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7. (2 points) Evaluate the following limit, if it exists. **You must use algebraic methods and justify your answer mathematically to earn credit for this question.**

$$\lim_{x\to 4^-} \frac{x^2 - x - 12}{|x-4|} = \lim_{x\to 4^-} \frac{(x-4)(x+3)}{|x-4|}$$

as $x \to 4$ from the left, $x-4$ will be negative $\therefore |x-4| = -(x-4)$

$$= \lim_{x\to 4^-} \frac{\cancel{(x-4)}(x+3)}{-\cancel{(x-4)}}$$

$$= \lim_{x\to 4^-} -(x+3)$$

$$= -(4+3)$$

$$= -7$$

Your answer: -7

8. (1 point) A study started tracking a herd of elk in northern Alberta. In January 1996, there were 1000 elk in the herd. Over the course of this study, scientists observed that the elk had a natural growth rate of 5% each year. Furthermore, in December each year, an average of 50 elk were killed by hunters. If h_t represents the herd's population (in January), t years after 1996, which of the following DTDS models the dynamics of this herd?

A. $h_{t+1} = 5h_t + 950$

B. $h_{t+1} = 50h_t - 1000$

[✓ C. $h_{t+1} = 1.05h_t - 50$]

D. $h_{t+1} = 1.05h_t + 950$

E. $h_{t+1} = 0.05h_t - 50$

F. $h_{t+1} = \frac{50}{1000}h_t + 0.05$

Your answer: C

1996: $h_0 = 1000$ elk

each year, last year's pop grows by 5% ⇒ slope $r = 1.05$

and 50 elk are killed ⇒ $d = -50$

linear $f(x) = rx + d$

$\therefore h_{t+1} = 1.05h_t - 50$

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9. (2 points) Find all solutions of the inequality. **Show your work and justify your reasoning.**

$$\frac{x-3}{x-5} < \frac{x}{x-1}$$

$$\Rightarrow \frac{x-3}{x-5} - \frac{x}{x-1} < 0$$

$$\Rightarrow \frac{(x-3)(x-1) - x(x-5)}{(x-5)(x-1)} < 0$$

$$\Rightarrow \frac{x^2-4x+3-x^2+5x}{(x-5)(x-1)} < 0$$

$$\Rightarrow \frac{x+3}{\underbrace{(x-5)(x-1)}} < 0$$

when is this − ?

	$(-\infty,-3)$	-3 $(-3,1)$	1 $(1,5)$	5 $(5,\infty)$
Sign of $x+3$	−	+	+	+
$x-1$	−	−	+	+
$x-5$	−	−	−	+
overall sign $\frac{x+3}{(x-5)(x-1)}$	−	+	−	+

Your answer: $(-\infty,-3) \cup (1,5)$

10. (1+1+2+1=5 points) A population of snails grows according to a Ricker model, so that the DTDS governing the growth of this population's density is given by

$$x_{t+1} = 2x_t e^{1-0.4x_t}$$

where t is in months and x_t denotes the number of snails per m^2 at time t.

(a) Give the updating function of this DTDS. $f(x) =$ $\boxed{2xe^{1-0.4x}}$

(b) Find all fixed points x^* of this DTDS. Show your work.

Solve $x = f(x)$

$x = 2xe^{1-0.4x}$

$0 = 2xe^{1-0.4x} - x$

$0 = x\left[2e^{1-0.4x} - 1\right]$

$x = 0$ $\qquad$ $2e^{1-0.4x} - 1 = 0$

$2e^{1-0.4x} = 1$

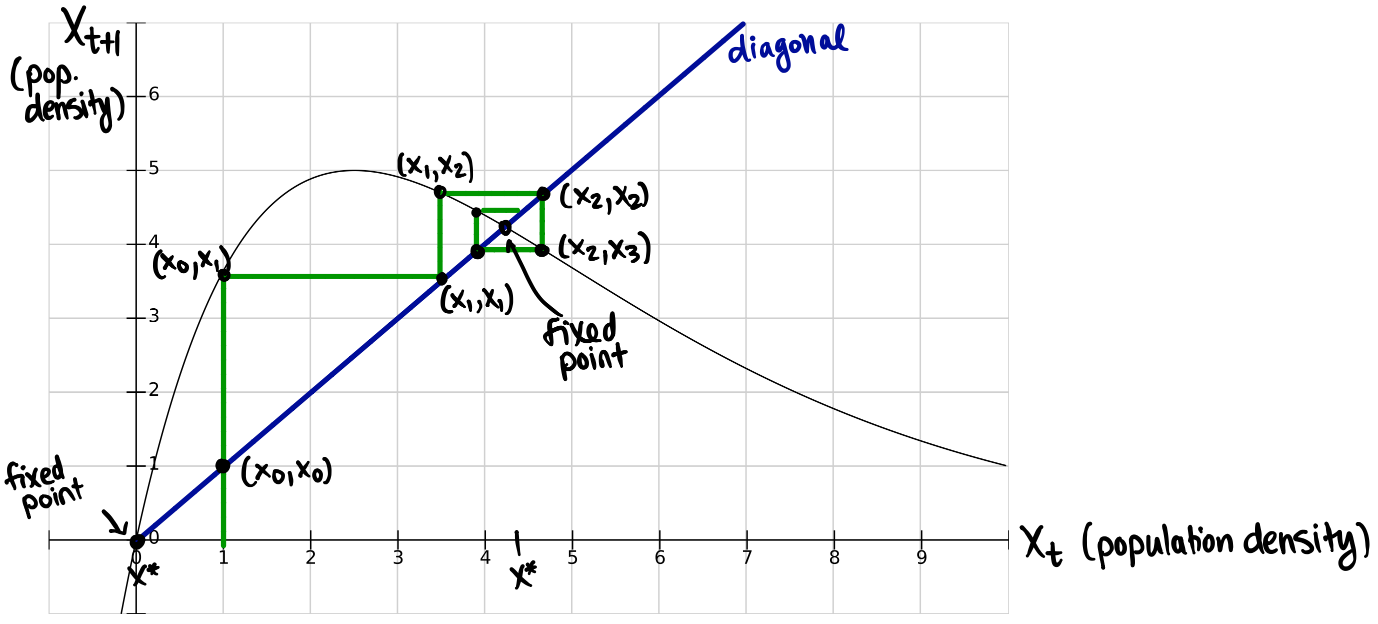
$e^{1-0.4x} = \frac{1}{2}$

$1 - 0.4x = \ln(\frac{1}{2})$

$x = \frac{\ln(1/2) - 1}{-0.4}$

Fixed points: $\boxed{x^* = 0 \text{ and } x^* = \frac{\ln 0.5 - 1}{-0.4} \approx 4.23}$

(c) The graph of $y = f(x)$ is drawn below. **Label the axes**, identify the **fixed points** of the DTDS on the graph, then **draw a cobweb** for at least three iterations, starting with an initial population density of $x_0 = 1$. *Use your student card as a straight edge.*



(d) What do you conclude will happen to this population of snails in the long term? (Your answer should be a complete sentence and should refer to the concepts of stability and fixed points as they pertain to this population of snails and your cobweb.)

Their density will approach (and eventually oscillate around) the stable fixed point density $x^* \approx 4.23$, and move away from the unstable fixed point $x^* = 0$.

11. (1+2+1=4 points) A patient is given a daily dose of the drug ToxicPlacebo, which is absorbed into the patient's system at a constant rate. A DTDS modeling the concentration of drug x_t, in mg/L in the patient's bloodstream each day just after receiving the dose is given by

$$x_{t+1} = 0.9x_t + 0.5, \quad \leftarrow \text{linear updating function}$$

$f(x) = mx + b$

$m = 0.9$ and $b = 0.5$

where t is measured in whole days since the treatment began.

(a) Find the fixed point of this DTDS, and give the general solution to this DTDS if the initial value is $\underline{x_0 = 1}$ mg/L.

fixed point $x^* = \frac{b}{1-m} = \frac{0.5}{1-0.9} = \frac{0.5}{0.1} = 5$

Fixed point: $x^* = \boxed{5}$

or solve $x = 0.9x + 0.5 \Rightarrow x^* = 5$

General solution: $\boxed{/_t} = \boxed{0.9^t(1-5)+5}$

general solution:

$x_t = m^t(x_0 - x^*) + x^*$

(b) If the initial value is $x_0 = 1$ mg/L, find the number of whole days it takes for the concentration of ToxicPlacebo in the patient's bloodstream to reach $\underline{\text{at least}}$ 4.5 mg/L.

we need to solve for t when $x_t \geqslant 4.5$ mg/L

Solve $0.9^t(1-5)+5 \geqslant 4.5$

$\Rightarrow 0.9^t(-4) \geqslant -0.5$

$\Rightarrow 0.9^t \leqslant \frac{-0.5}{-4}$

$\Rightarrow \ln(0.9^t) \leqslant \ln(0.125)$

$\Rightarrow t\ln(0.9) \leqslant \ln(0.125)$

$\Rightarrow t \geqslant \frac{\ln(0.125)}{\ln(0.9)}$

Minimum number of whole days required: $\boxed{\text{20 days}}$

$\Rightarrow t \geqslant \frac{\ln(0.125)}{\ln(0.9)} \approx 19.736\ldots$

(c) Surprisingly, testing over the year reveals that the concentration of ToxicPlacebo in the patient's bloodstream has stabilized to $x^* = 6$ mg/L. An investigation reveals that the doctor's bad handwriting was to blame, and the administered daily dose d in reality was not actually 0.5 mg/L. What was it?

If actual fixed point is $x^* = 6$ mg/L, then $f(x) = 0.9x + d$

must satisfy $x^* = f(x^*) \Rightarrow 6 = 0.9(6) + d \Rightarrow d = 0.6$ mg/L

Alternate solution: $x^* = \frac{d}{1-r}$ so $6 = \frac{d}{1-0.9} \Rightarrow d = 6(0.1) = 0.6$ mg/L

$\therefore$ Actual daily dose has been 0.6mg/L.