Version A

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Discrete Mathematics for Computing MAT1348B

First Midterm Examination

8 February 2016
Prof. Philip J. Scott

Instructions. *You must sign below to confirm that you have read, understand, and will follow them.*

- This is an 80-minute *closed-book* exam; no notes are allowed. Calculators are *not* permitted.
- The exam consists of 11 questions on 10 pages. Page 11 contains a table of logical equivalences for your convenience.
- Questions 1-6 are short-answer. Write the final answer in the appropriate answer box, and briefly justify your answer where required.
- Questions 7-8 are true-false. In each part, you must circle the correct response. You need not justify your answers.
- Questions 9-11 are long-answer. To receive full marks, your solution/proof must be complete, correct, and show all relevant details.
- Be sure to read carefully and follow the instructions for the individual problems.
- For rough work, you may use the back pages. Do not use scrap paper of your own.
- Use proper mathematical notation and terminology.
- If you require clarification, raise your hand.
- Unauthorized electronic devices (such as cellular phones) are not permitted during this exam. Such devices must be turned off completely and stored out of students' reach. Students found in possession of such a device during the exam will be asked to leave immediately and academic fraud allegations may be filed.

LAST NAME: Solutions

First name: __________

Signature: __________

LAST NAME: Solutions

First name:

Student number:

Question	1 – 2	3 – 4	5	6	7 – 8	9	10	11	Total
Max	$2+2$	$3+2$	4	4	$3+4$	5	6	5	40
Marks									

Short-answer questions — write your final answer in the answer box. Wherever indicated, you must briefly justify your answers to receive full marks.

[2pts] 1. The truth table of a compound proposition P with atomic propositions A, B, and C is as follows:

A	B	C	P	
T	T	T	F	
T	T	F	T	$\leftarrow$ $A \land B \land \neg C$
T	F	T	F	
T	F	F	T	$\leftarrow$ $A \land \neg B \land \neg C$
F	T	T	F	
F	T	F	T	$\leftarrow$ $\neg A \land B \land \neg C$
F	F	T	F	
F	F	F	F	

Give a **disjunctive normal form** of P. *Do not simplify your answer.*

DNF: $(A \land B \land \neg C) \lor (A \land \neg B \land \neg C) \lor (\neg A \land B \land \neg C)$

No justification is needed.

[2pts] 2. Complete the following definition:

A set $\{P_1, P_2, P_3\}$ is said to be **consistent** if for some valuation V, $V(P_1) = V(P_2) = V(P_3) = T$. That is, some row of the truth table fr $(P_1 \land P_2 \land P_3)$ gives the value T — i.e. the formulas P_1, P_2, P_3 are simultaneously true for some valuation.

No justification is needed.

[3pts] 3. Give an example of a proposition in propositional variables a and b that is:

- a tautology: $(a \vee \neg a) \vee b$
- a contradiction: $(a \wedge \neg a) \wedge b$
- a contingency $a \vee b$

There are many possible answers

No justification is needed.

[2pts] 4. Consider the following atomic propositions:

N: “The system is operating normally.”

M: ”Messages are scanned for viruses.”

U: ”Software is being updated.”

F: “Users can access the file system. ”

Translate the following sentence into a compound proposition using propositional variables N, M, U, and F. *Parentheses are included to clarify the structure.*

(**(The system operating normally and software not being updated) is necessary and sufficient for users to be able to access the file system**)
only if messages are not being scanned for viruses.

Compound proposition: $((N \wedge \neg U) \leftrightarrow F) \longrightarrow \neg M$

No justification is needed.

[4pts] 5. On the Island of Knights and Knaves, as you know, there are two types of natives, indistinguishable by sight: knights, who always tell the truth, and knaves, who always lie.

Strolling on the island, we meet two inhabitants A and B. Person A says: "Both of us are knaves." What can you conclude about the types (knight or knave) of persons A and B?

Answer: A Knave, B Knight.

Justification (write solutions in words–no truth tables. Be precise and clear):

Case 1: A is a Knight. ∴ A speaks T. But A says they are both knaves. ✗ Contradiction.

Case 2: A is a Knave. ∴ A lies. But A says "A Knave ∧ B Knave"
∴ Since "A knave" is actually True, for the entire statement to be false, "B knave" is false, so B must be a Knight. Consistent!

∴ A is a Knave, B is a Knight.

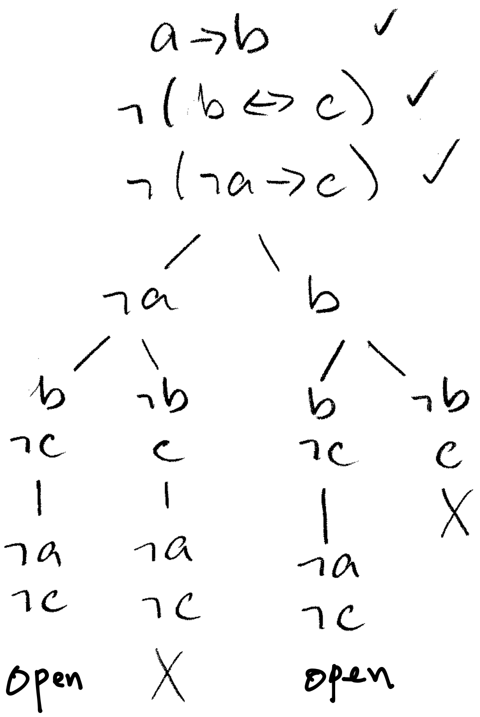
[4pts] 6. Is the following argument valid? If not, give a counterexample.

$$\begin{array}{l} a \to b \\ \underline{\neg(b \leftrightarrow c)} \\ \therefore \neg a \to c \end{array}$$

Answer: The argument is (circle): valid [✓ invalid]

Counterexample (if applicable): a=F, b=T, c=F

Justification:



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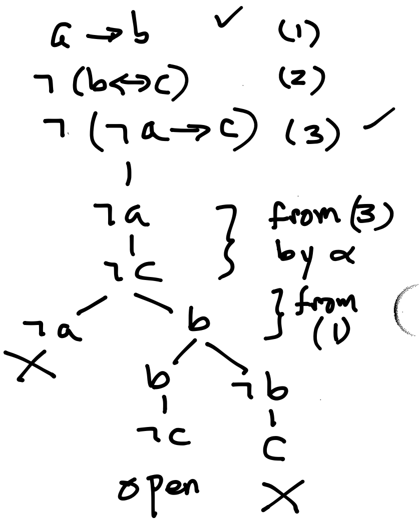


Or:

a	b	c	a→b	¬(b⇔c)	¬a→c	
T	T	T	T	F	T	
T	T	F	T	T	T	
T	F	T	F	T	T	
T	F	F	F	F	T	
F	T	T	T	F	T	
F	T	F	T	T	F	← counterexample
F	F	T	T	T	T	
F	F	F	T	F	F	

True-false questions. Circle the correct response. No justification needed.

7. Consider an argument with premises P_1 and P_2, and conclusion C:

$$\begin{array}{r} P_1 \\ \underline{P_2} \\ \therefore C \end{array}$$

[3pts] Which of the following statements about the above argument are true and which are false?

(a) If $P_1 \wedge P_2 \wedge \neg C$ is a contradiction, then the argument is valid. [✓ True] False

(b) If $(P_1 \vee P_2) \rightarrow C$ is a tautology, then the argument is valid. [✓ True] False

(c) If the argument is valid, then $\{P_1, P_2, C\}$ is consistent. True [✓ False]

8. Let P be a complex proposition, and consider a complete truth tree with P at the root.
[4pts] Answer the following as true or false.

(a) If the truth tree for P has no closed branches, then P is a tautology.
True [✓ False]

(b) If the truth tree for P has no open branches, then $\neg P$ is a tautology.
[✓ True] False

(c) The number of complete open branches is equal to the number of counterexamples to the statement "P is a contradiction".
True [✓ False]

(d) Each complete open branch corresponds to one or more counterexamples to the statement "$\neg P$ is a tautology".
[✓ True] False

Recall $\begin{array}{c} P_1 \\ P_2 \\ \hline \therefore C \end{array}$ is valid iff $((P_1 \wedge P_2) \to C)$ is a tautology.

7 (a) If $P_1 \wedge P_2 \wedge \neg C \equiv F$, then $((P_1 \wedge P_2) \wedge \neg C) \longrightarrow F$ is a tautology

so $(P_1 \wedge P_2) \to (\neg C \to F)$ " " "

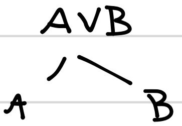
so $(P_1 \wedge P_2) \to C$ " "

∴ The argument is valid.

(b) Suppose $(P_1 \vee P_2) \to C$ is true. We must show $(P_1 \wedge P_2) \to C$ is true. Suppose $P_1 \wedge P_2$ is true. Then (say) P_1 is true. So $(P_1 \vee P_2)$ is true. But $(P_1 \vee P_2) \to C$ is true. Hence C is true, by modus ponens. ∴ $(P_1 \wedge P_2) \to C$ is true.

(c) False. If $P_1 = F$, $(P_1 \wedge P_2) \to C$ is a tautology, so the argument is valid, but $\{P_1, P_2, C\}$ is inconsistent.

⑧ (a)



has no closed branches but $A \vee B$ is not a tautology. ∴ False.

(b) To check if $\neg p$ is a tautology, put $\neg\neg p$ (i.e. p) at the top of the tree. If all branches are closed, it's a tautology. ∴ True.

(c) The number of complete open branches ≠ the number of valuations, since 2 or more open branches might give the same valuation. ∴ False

(d) In (c) above, if there are open branches in the tree, each open branch is a valuation making p true (i.e. $\neg p$ false). So is certainly a counterexample to "$\neg p$ is a tautology".

∴ True!

Long-answer questions. Detailed solutions are required.

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[~~4~~pts] 9. Prove the following using only the equivalences listed on Page 11. Justify each step by giving the number of the corresponding equivalence on Page 11. *Do not skip steps or combine several equivalences into a single step. Do not omit parentheses.*

$$(a \vee b) \wedge \Big((a \to c) \wedge (b \to c)\Big) \equiv (a \vee b) \wedge c$$

$(a\vee b)\wedge((a\to c)\wedge(b\to c))$ (1)

$\equiv (a\vee b)\wedge((\neg a\vee c)\wedge(\neg b\vee c))$ (13)

$\equiv (a\vee b)\wedge((c\vee\neg a)\wedge(c\vee\neg b))$ (17)

$\equiv (a\vee b)\wedge(c\vee(\neg a\wedge\neg b))$ (20)

$\equiv (a\vee b)\wedge(c\vee\neg(a\vee b))$

(18)

$\equiv ((a\vee b)\wedge c)\vee((a\vee b)\wedge\neg(a\vee b))$

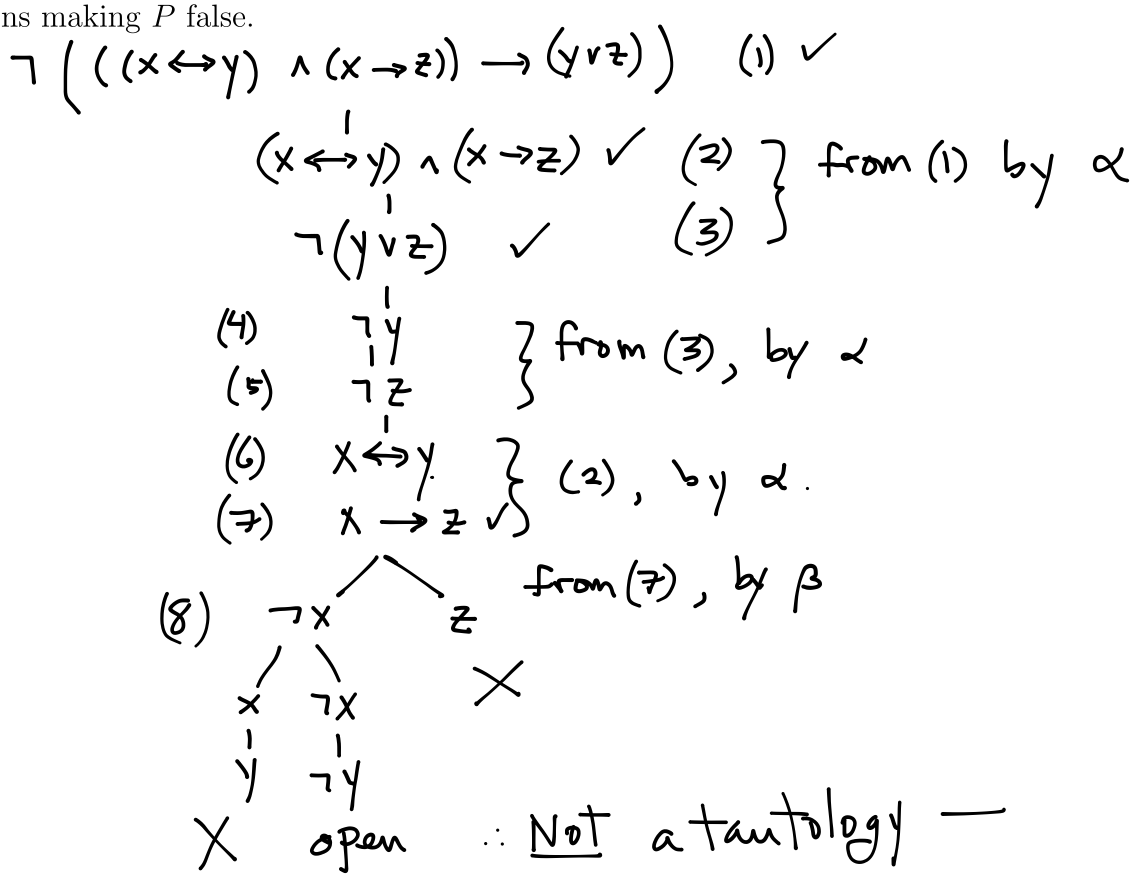
(5)

$\equiv ((a\vee b)\wedge c)\vee \mathbf{F}$

(6)

$\equiv (a\vee b)\wedge c$

[6pts] 10. Let P be the proposition $((x \leftrightarrow y) \wedge (x \rightarrow z)) \rightarrow (y \vee z)$. Using an **appropriate truth tree**, determine whether or not P is a tautology. If the answer is negative, give all valuations making P false.



∴ Not a tautology —

$V(x) = V(y) = V(z) = F$ falsifies the formula.

Answer: The proposition is a tautology (circle): True [✓ False]

All valuations making P false (if applicable): $V(x) = V(y) = V(z) = F$.

11. *Recall that an integer m divides an integer n if $n = km$ for some integer k.*

[5pts]

Let m and n be positive integers. Give an **indirect proof** of the following theorem. *(Parentheses are included to clarify the structure of the statement.)*

Theorem: If n is odd, then [m is odd or m does not divide n].

Strategy: Define p: "n is odd"

q: "m is odd"

r: "m divides n"

We have to prove $p \to (q \vee \neg r)$ using an indirect proof. Hence, we assume $\neg(q \vee \neg r)$ and show that $\neg p$ follows. Note that $\neg(q \vee \neg r) \equiv \neg q \wedge r$.

Proof. Assume m is even and m divides n.

Hence $m = 2k$ for some integer k, and

$n = lm$ for some integer l. Then:

$n = lm = l(2k) = 2(lk) = 2t$ for $t = lk$.

Since l and k are integers, so is t.

Hence n is even.

Conclusion: We proved $(\neg q \wedge r) \to \neg p$ directly. This gives an indirect proof of $p \to (q \vee \neg r)$

Table of logical equivalences. *You may detach this page.*

(Note: I added the Domination Laws (8), (9) below, to simplify your calculations)

	Equivalence	Name
(1)	$P \to Q \equiv \neg P \vee Q$	
(2)	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	
(3)	$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$	
(4)	$P \vee \neg P \equiv \mathbf{T}$	Negation Laws
(5)	$P \wedge \neg P \equiv \mathbf{F}$	
(6)	$P \vee \mathbf{F} \equiv P$	Identity/Unit Laws
(7)	$P \wedge \mathbf{T} \equiv P$	
(8)	$P \vee \mathbf{T} \equiv \mathbf{T}$	Domination Laws
(9)	$P \wedge \mathbf{F} \equiv \mathbf{F}$	
(10)	$P \vee P \equiv P$	Idempotent Laws
(11)	$P \wedge P \equiv P$	
(12)	$\neg\neg P \equiv P$	Double negation
(13)	$P \vee Q \equiv Q \vee P$	Commutative Laws
(14)	$P \wedge Q \equiv Q \wedge P$	
(15)	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	Associative Laws
(16)	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	
(17)	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	Distributive Laws
(18)	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	
(19)	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	De Morgan's Laws
(20)	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	