PART A (35 marks)

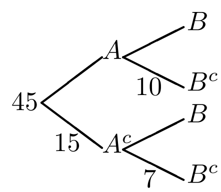
A1. Let $U = \{a, b, c, d, e, f, g, h\}$, $A = \{b, d, f, h\}$, $B = \{a, b, c, d\}$ and $C = \{a, b, d, e, g\}$.
Find $(A \cap B) \cup C^c$.

A: $\{b, f, h\}$	B: $\{c, f, h\}$	C: $\{a, b, d, e, g\}$	D: $\{b, c, d, f, h\}$	E: $\{a, b, c, d, e, f, g, h\}$

Solution: We have $A \cap B = \{b, d, f, h\} \cap \{a, b, c, d\} = \{b, d\}$ and $C^c = \{c, f, h\}$, so we get

$$(A \cap B) \cup C^c = \{b, d\} \cup \{c, f, h\} = \{b, c, d, f, h\}$$

A2. Consider the counting tree shown below. Find $n(A \cup B)$.



A: 8	B: 20	C: 28	D: 30	E: 38

Solution: The easiest way to find $n(A \cup B)$ is to add to either $n(A)$ or $n(B)$ the only part of $A \cup B$ which is missing, which is the number in the part of the other set that isn't in the set we've already counted. Finding $n(A)$ is easier than finding $n(B)$ here. Since $n(U) = 45$ and $n(A^c) = 15$, we see that $n(A) = 45 - 15 = 30$. This already includes the elements which are in $A \cap B$, so we just need to include the elements of B which are not in A, i.e. the elements of $A^c \cap B$. Since $n(A^c) = 15$ and $n(A^c \cap B^c) = 7$, then $n(A^c \cap B) = 15 - 7 = 8$. Therefore we have

$$n(A \cup B) = n(A) + n(A^c \cap B) = 30 + 8 = 38$$

Of course, you may have done this the other way, using the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. To use that formula, you would need to find $n(A \cap B)$, which is $n(A \cap B) = n(A) - n(A \cap B^c) = 30 - 10 = 20$. Now that you know that number, you add it to $n(A^c \cap B) = 15 - 7 = 8$ to get $n(B)$, and then subtract it from the sum $n(A) + n(B)$, because it's now been included twice, once in $n(A)$ and once in $n(B)$.

A3. One day, all of the 80 cars parked in the Middlesex College parking lot were either SUVs or sedans. 30 of the SUVs did not have a sunroof, and 10 of the cars which did have a sunroof were sedans. If the number of SUVs with a sunroof was the same as the number of sedans without a sunroof, how many SUVs were there?

A: 30	B: 40	C: 50	D: 60	E: 70

Solution: The universal set is all of the cars in the parking lot, with $n(U) = 80$. Let A be the set of all SUVs in the lot. Since a sedan and an SUV are different kinds of cars, and these are the only kind of cars in the lot, A^c is the set of all sedans in the lot. Let B be the set of all the cars in the lot which have a sunroof. Then we know that $n(A \cap B^c) = 30$ and $n(A^c \cap B) = 10$. Also, since there are the same number of SUVs with a sunroof as sedans without a sunroof, we have $n(A \cap B) = n(A^c \cap B^c)$. Let x be this unknown number. Since $n(A) + n(A^c) = n(U)$, and $n(A) = n(A \cap B) + n(A \cap B^c)$, while $n(A^c) = n(A^c \cap B) + n(A^c \cap B^c)$, then

$$n(U) = n(A \cap B) + n(A \cap B^c) + n(A^c \cap B) + n(A^c \cap B^c) = x + 30 + 10 + x = 2x + 40$$

and knowing that $n(U) = 80$ we get $2x + 40 = 80$, so $2x = 40$ and $x = 20$. So the number of SUVs is

$$n(A) = n(A \cap B) + n(A \cap B^c) = x + 30 = 20 + 30 = 50$$

Note: You may have used a counting tree. If so, you did (or should have done) pretty much the same calculations as shown here.

A4. How many *non-empty* subsets of the set $\{1, 2, 3, 4, 5, 6\}$ either contain only odd numbers or contain only even numbers?

A: 7	B: 8	C: 14	D: 15	E: 16

Solution: Let's call the given set S. Any subset of S which contains only odd numbers must actually be a subset of $\{1, 3, 5\}$, which has $2^3 = 8$ subsets. And any subset of S which contains only even numbers must actually be a subset of $\{2, 4, 6\}$, which also has $2^3 = 8$ subsets. But in each of these counts, we have included the empty set, which is a subset of both $\{1, 3, 5\}$ and $\{2, 4, 6\}$, but does not contain "only odd numbers" or "only even numbers". So there are only 7 subsets of S which do contain only odd numbers, and likewise only 7 subsets of S which do contain only even numbers, for a total of 14 subsets which either contain only odd numbers or contain only even numbers.

A5. Western Science is holding a job fair in May. There are 2 banks, 3 companies from the industry sector and 5 programming companies registered to attend the fair. In how many ways can the 10 booths at the fair be arranged in a circle in the Physics building atrium so that all companies of the same type are together, side-by-side, in the circle?

A: $2!3!5!$	B: $2!2!3!5!$	C: $4!1!2!$	D: $9!$	E: $\dfrac{2!3!5!}{10}$

Solution: There are 3 types of companies, and we want to arrange these 3 types in a circle. There are $(3-1)! = 2!$ ways to do that. That is, there are 2! ways to arrange 10 booths in a circle with 2 side-by-side designated to be for the banks, 3 side-by-side designated to be for the companies from the industry sector, and the remaining 5, also side-by-side, designated to be for the programming companies. But now, we must arrange the companies within each type. There are 2! ways to arrange the 2 banks (at their 2 booths), 3! ways to arrange the 3 companies from the industry sector, and 5! ways to arrange the 5 programming companies. So there are 2!2!3!5! ways to arrange the 10 booths with companies of the same type together.

A6. There are 7 volunteers who will work at the Western Teaching Conference this summer. In how many ways can the organizer of the conference divide these volunteers into 2 groups of 2 volunteers and one group of 3 volunteers?

A: $\binom{7}{2\ 2\ 3}$	B: $\binom{7}{2\ 2\ 3} \div 3!$	C: $\binom{7}{2\ 2\ 3} \times 3!$	D: $\binom{7}{2\ 2\ 3} \times 2!$	E: $\binom{7}{2\ 2\ 3} \div 2!$

Solution: Since we don't know what the volunteers will be doing, the 2 groups of 2 volunteers are indistinguishable, i.e. are the same size of group and will be doing the same thing (just "being in a group together"). But the group of 3 volunteers is distinguishable from the other 2 groups, because that group contains 3 people, not 2. And the number of ways to divide 7 objects into 2 indistinguishable groups of 2 and 1 group of 3 is

$$\binom{7}{2\ 2\ 3} \div 2!$$

A7. In how many ways can all of 6 identical notebooks be distributed among 10 students if each student can receive at most one notebook?

A: 1	B: $\binom{10}{6}$	C: $\binom{15}{6}$	D: $\binom{15}{10}$	E: 2^{10}

Solution: There are 10 students and only 6 notebooks. If each student is to receive at most one notebook, then each must get 0 or 1 notebook. Since the notebooks are identical, it doesn't matter *which* notebook a student gets, only whether or not the student gets one. So we simply need to choose which 6 of the 10 students get a notebook. The number of ways to choose 6 of the 10 students is $\binom{10}{6}$.

A8. In how many ways can all of 10 identical notebooks be distributed among 6 students if each student must receive at least one notebook?

A: 1	B: $\binom{15}{10}$	C: $\binom{15}{6}$	D: $\binom{9}{4}$	E: $\binom{10}{6}$

Solution: This time, there are 10 notebooks and only 6 students. We are told that each student must receive at least one notebook, so we start by giving each of them one. Again, the notebooks are all identical, so it doesn't matter which one a student gets. Now, the 4 remaining notebooks could be distributed in any way among the 6 students. Therefore what we have now is a *free distributions* problem, so we count the number of ways of distributing $k = 4$ identical objects among $r = 6$ people:

$$\binom{k+r-1}{k} = \binom{4+6-1}{4} = \binom{9}{4}$$

A9. Consider the experiment *draw 2 cards from a standard deck of 52 cards.* Which of the following are sample spaces for this experiment?

$$S_1 = \{\text{both cards are the same colour, the cards are different suits}\}$$
$$S_2 = \{\text{both cards are Clubs, both cards are Spades, both cards are red}\}$$
$$S_3 = \{\text{both cards are red, both cards are black, the cards are different colours}\}$$

A: S_1 and S_3 only.	B: S_1 and S_2 only.	C: S_2 only.	D: S_3 only.	E: None of A, B, C or D.

Solution: In order to be a sample space, the descriptions in a set must not overlap, and must include all the things that could happen when the experiment is performed. S_1 is not a sample space, since the 2 descriptions overlap: if one card is a Heart and the other is a Diamond, the cards are different suits and they are also both the same colour (red). And S_2 is not a sample space because it does not include all possibilities. For instance, if one Club and one Heart are drawn, then none of the descriptions applies because Clubs are black. But S_3 *is* a sample space. There are only two colours of cards in the deck, so when 2 cards are drawn, they can only be both red, both black, or different colours. And these possibilities don't overlap, since each card is only one colour.

A10. Consider the experiment *draw 2 balls from a box containing 10 red balls and 5 white balls.* Which of the following are *equiprobable* sample spaces for this experiment?

$$S_1 = \{\text{both balls are the same colour, the balls are different colours}\}$$
$$S_2 = \{\text{both balls are red, both balls are white, the balls are different colours}\}$$
$$S_3 = \{\text{more red than white, more white than red}\}$$

A: S_1 and S_2 only.	B: S_1 and S_3 only.	C: S_1 only.	D: S_2 only.	E: None of S_1, S_2 or S_3.

Solution: In order to be an equiprobable sample space, a set must first of all *be* a sample space, and then must also have the characteristic that each description is equally likely to occur. Set S_3 is not a sample space, since it does not include the possibility of drawing one ball of each colour. Therefore it cannot be an equiprobable sample space. The other two sets are both sample spaces, so we need to determine whether each is equiprobable. We know that there are 10 red balls and 5 white balls. To draw 2 balls of different colours, we could choose any one of the 10 red balls and choose any one of the 5 white balls, so there are $10 \times 5 = 50$ ways that this could happen. To draw 2 balls the same colour, we could choose any 2 of the 10 red balls or choose any 2 of the 5 white balls. There are $\binom{10}{2} = 45$ ways to choose 2 red balls and

$\binom{5}{2} = 10$ ways to choose two white balls, so there are $45 + 10 = 55$ ways to choose 2 balls the same colour. Therefore choosing 2 the same colour is more likely to happen than choosing balls of different colours, so S_1 is not equiprobable. And neither is S_2, since we have already seen that there are more ways to choose one of each colour than to choose 2 red balls (and more ways to choose 2 red than 2 white). So none of the given sets is an equiprobable sample space.

A11. Three girls and four boys are arranged in a line at random. What is the probability that the boys are all together if it is known that the girls are all together?

A: $\frac{1}{5}$	B: $\frac{2}{5}$	C: $\frac{4}{35}$	D: $\frac{1}{7}$	E: $\frac{1}{2}$

Solution: Let S be the set of all ways in which the 3 girls and 4 boys can be arranged in a line. Then since there are 7 people being arranged in the line, and it is done at random, S is an equiprobable sample space with $n(S) = 7!$. Let E be the event that the boys are all together and F be the event that the girls are all together. Then what we have been asked to find is $Pr[E \mid F]$. To count the number of ways of arranging 3 girls and 4 boys in a line with the girls all together, we think of arranging one group of girls and 4 individual boys in a line (5 objects, so this can be done in 5! ways) and then arranging the 3 girls within their group (which can be done in 3! ways). This gives $n(F) = 5!3!$. And to count the number of ways of arranging the 7 people in a line so that all of the girls are together *and* all of the boys are together, we count the number of ways to arrange one group of girls and one group of boys in a line (2 objects, so 2! ways) and then the number of ways to arrange the 3 girls within their group (3! ways) and to arrange the 4 boys within their group (4! ways), so we see that $n(E \cap F) = 2!3!4!$. Therefore we get

$$Pr[E \mid F] = \frac{Pr[E \cap F]}{Pr[F]} = \frac{n(E \cap F)}{n(F)} = \frac{2!3!4!}{5!3!} = \frac{2!4!}{5 \times 4!} = \frac{2}{5}$$

A12. Let E and F be two independent events defined on the same sample space. If $Pr[E] = 0.2$ and $Pr[F] = 0.4$, find $Pr[E^c \cap F^c]$.

A: 0.32	B: 0.4	C: 0.48	D: 0.52	E: 0.6

Solution: Since E and F are independent events, then $Pr[E \cap F] = Pr[E] \times Pr[F] = (0.2)(0.4) = 0.08$. We use this to see that

$$Pr[E \cap F^c] = Pr[E] - Pr[E \cap F] = 0.20 - 0.08 = 0.12$$

And now, we use this to find $Pr[E^c \cap F^c]$:

$$Pr[E^c \cap F^c] = Pr[F^c] - Pr[E \cap F^c] = (1 - Pr[F]) - Pr[E \cap F^c] = (1 - 0.4) - 0.12 = 0.6 - 0.12 = 0.48$$

A13. Let E and F be two mutually exclusive events defined on the same sample space. If $Pr[E] = 0.2$ and $Pr[F] = 0.4$, find $Pr[E^c \cap F^c]$.

A: 0.32	B: 0.4	C: 0.48	D: 0.52	E: 0.6

Solution: Since E and F are mutually exclusive, then $Pr[E \cap F] = 0$. So $Pr[E \cap F^c] = Pr[E] = 0.2$ and

$$Pr[E^c \cap F^c] = Pr[F^c] - Pr[E \cap F^c] = (1 - 0.4) - 0.2 = 0.6 - 0.2 = 0.4$$

A14. Tom sleeps through his morning alarm on $\frac{1}{4}$ of school days. When he sleeps through the alarm, $\frac{3}{4}$ of the time he misses the school bus, but when he doesn't sleep through the alarm, he only misses the school bus $\frac{1}{8}$ of the time. When Tom misses the school bus, he has to walk to school. What is the probability that Tom will have to walk to school on a typical school day?

A: $\dfrac{9}{32}$	B: $\dfrac{23}{32}$	C: $\dfrac{3}{16}$	D: $\dfrac{3}{4}$	E: $\dfrac{1}{4}$

Solution: Let S be the event that Tom sleeps through his alarm, and M be the event that he misses the school bus. Then we are told that $Pr[S] = \frac{1}{4}$, so $Pr[S^c] = \frac{3}{4}$ is the probability that he doesn't sleep through the alarm. And we are told that $Pr[M \mid S] = \frac{3}{4}$, so that $Pr[M^c \mid S] = \frac{1}{4}$, as well as that $Pr[M \mid S^c] = \frac{1}{8}$, so that $Pr[M^c \mid S^c] = \frac{7}{8}$. Now, we are asked for the probability that Tom will have to walk to school, which happens when he misses the school bus, so we need to find $Pr[M]$. We get

$$\begin{aligned} Pr[M] &= Pr[M \cap S] + Pr[M \cap S^c] = (Pr[M \mid S] \times Pr[S]) + (Pr[M \mid S^c] \times Pr[S^c]) \\ &= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{8}\right)\left(\frac{3}{4}\right) = \frac{3}{16} + \frac{3}{32} = \frac{6}{32} + \frac{3}{32} = \frac{9}{32} \end{aligned}$$

A15. See question A14. Last Monday (a school day), Tom did not miss the school bus. What is the probability that he *did not* sleep through his morning alarm that day?

A: $\dfrac{3}{4}$	B: $\left(\dfrac{3}{4} \times \dfrac{7}{8}\right) + \left(\dfrac{1}{4} \times \dfrac{1}{4}\right)$	C: $\dfrac{3}{4} \times \dfrac{7}{8}$	D: $\dfrac{\frac{3}{4} \times \frac{7}{8}}{\left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{3}{4} \times \frac{7}{8}\right)}$	E: $\dfrac{\frac{1}{4} \times \frac{1}{4}}{\left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{3}{4} \times \frac{7}{8}\right)}$

Solution: Now we are asked for $Pr[S^c \mid M^c]$:

$$Pr[S^c \mid M^c] = \frac{Pr[S^c \cap M^c]}{Pr[M^c]} = \frac{Pr[M^c \mid S^c] \times Pr[S^c]}{(Pr[M^c \mid S] \times Pr[S]) + (Pr[M^c \mid S^c] \times Pr[S^c])} = \frac{\frac{7}{8} \times \frac{3}{4}}{\left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{7}{8} \times \frac{3}{4}\right)}$$

(Of course, you probably drew a probability tree in question A14 and used Bayes' formula here.)

A16. Ten separate full decks of cards are in a row on a table. One card is drawn from each deck. What is the probability that exactly 2 of the drawn cards are black and exactly 3 of them are Hearts?

A: $\binom{10}{2\ 3\ 5}\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right)^3\left(\frac{1}{4}\right)^5$	B: $\binom{10}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right)^3$	C: $\binom{10}{2\ 3\ 5}\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right)^3$
D: $\binom{10}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right)^3\left(\frac{1}{4}\right)^5$	E: $\binom{10}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^8 \times \binom{10}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^7$	

Solution: Drawing one card from each of 10 separate decks corresponds to performing 10 independent trials of the experiment "draw one card from a full deck". In this experiment, let B be the event that a black card is drawn, and H be the event that at Heart is drawn. Then, since half of the cards (the Spades and Clubs) are black, and one-quarter of them are Hearts, we have $p_B = Pr[B] = \frac{1}{2}$ and $p_H = Pr[H] = \frac{1}{4}$. Of course, there is one other thing that could happen: let D be the event that a Diamond (i.e. a card which is neither black nor Hearts) is drawn, with $p_D = Pr[D] = \frac{1}{4}$. Then the probability that during the $n = 10$ trials, exactly $n_B = 2$ black cards and exactly $n_H = 3$ Hearts are drawn, which means that there must also have been exactly $n_D = 5$ Diamonds drawn, is given by

$$\binom{n}{n_B\ \ n_H\ \ n_D}(p_B)^{n_B}(p_H)^{n_H}(p_D)^{n_D} = \binom{10}{2\ 3\ 5}\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right)^3\left(\frac{1}{4}\right)^5$$

A17. If X is a Binomial random variable counting the number of successes in 30 Bernoulli trials with probability of success 0.65, find the probability that the value of X is at least 2.

A: $\binom{30}{2}(0.65)^2(0.35)^{28}$	B: $1-\binom{30}{2}(0.65)^2(0.35)^{28}$	C: $1-\binom{30}{1}(0.65)^1(0.35)^{29}$
D: $(0.35)^{30}+\binom{30}{1}(0.65)^1(0.35)^{29}$	E: $1-\left[(0.35)^{30}+\binom{30}{1}(0.65)^1(0.35)^{29}\right]$	

Solution: We have $X = B(30, 0.65)$, so the probability that $X = x$ (i.e. that exactly x successes are observed) is given by the Bernoulli formula as

$$Pr[X = x] = \binom{n}{x}(0.65)^x(0.35)^{n-x} = \binom{30}{x}(0.65)^x(0.35)^{30-x}$$

Let E be the event that at least 2 successes are observed. Then E occurs whenever there are *not* exactly 0 or exactly 1 success observed. That is $E^c = (X = 0) \cup (X = 1)$, so we see that

$$\begin{aligned} Pr[E^c] &= Pr[X = 0] + Pr[X = 1] = \binom{30}{0}(0.65)^0(0.35)^{30} + \binom{30}{1}(0.65)^1(0.35)^{29} \\ &= 1 \times 1 \times (0.35)^{30} + \binom{30}{1}(0.65)^1(.35)^{29} \\ \Rightarrow Pr[X \geq 2] &= Pr[E] = 1 - \left[(0.35)^{30} + \binom{30}{1}(0.65)^1(0.35)^{29}\right] \end{aligned}$$

A18. X is a discrete random variable whose probability distribution function (pdf) is shown below. What is the value of $Pr[X \geq 0]$?

x	$Pr[X = x]$
-10	0.1
-2	0.4
0	p
8	$2p$
15	0.2

A: 0.1	B: 0.2	C: 0.3	D: 0.4	E: 0.5

Solution:

$$Pr[X \geq 0] = Pr[(X < 0)^c] = 1 - Pr[X < 0] = 1 - (Pr[X = -10] + Pr[X = -2]) = 1 - (0.1 + 0.4) = 1 - 0.5 = 0.5$$

On the other hand, you may have found $Pr[X \geq 0]$ directly, which requires finding p. Since the probabilities must sum to 1, we find that $3p + 0.7 = 1$, which gives $p = 0.1$ so that $Pr[X \geq 0] = 0.1 + 0.2 + 0.2 = 0.5$.

A19. X is a discrete random variable whose probability distribution function (pdf) is shown below. If the expected value of X is $E(X) = -2$, find the value of a.

x	$Pr[X = x]$
a	0.3
0	b
10	0.1

A: -1	B: -2	C: -10	D: 10	E: 0.6

Solution: We know that

$$E(X) = a \times Pr[X = a] + 0 \times Pr[X = 0] + 10 \times Pr[X = 10] = a(0.3) + 0(b) + 10(0.1) = 0.3a + 1$$

So knowing that $E(X) = -2$ tells us that

$$0.3a + 1 = -2 \Rightarrow 0.3a = -3 \Rightarrow a = \frac{-3}{0.3} = -\frac{3}{3/10} = -3 \times \frac{10}{3} = -10$$

A20. Juan has chosen one of the 5 vowels (a, e, i, o or u) at random, and Connie is going to try to guess which one he chose. If she is correct, Juan will pay her \$100, but if she guesses wrong, she must pay Juan \$20. What is the expected value of Connie's winnings?

A: \$0	B: \$2	C: \$4	D: \$20	E: \$80

Solution: Connie has a 1 in 5 chance of guessing the vowel that Juan chose, so the probability that she is right is $\frac{1}{5}$ and the probability that she is wrong is $\frac{4}{5}$. Letting X be the amount (in dollars) that Connie wins, we have $Pr[X = 100] = \frac{1}{5}$ and $Pr[X = -20] = \frac{4}{5}$, so we see that the expected value of X is

$$E(X) = 100 \times Pr[X = 100] + (-20) \times Pr[X = -20] = 100 \times \frac{1}{5} - 20 \times \frac{4}{5} = 20 - 16 = 4$$

That is, since her winnings are measured in dollars, the expected value of Connie's winnings is \$4.

A21. Let X be a discrete random variable, with $Pr[X = -4] = \frac{1}{2}$, $Pr[X = 0] = \frac{1}{4}$ and $Pr[X = 4] = \frac{1}{4}$, so that $\mu = -1$. Find $V(X)$.

A: 11	B: -5	C: 0	D: 12	E: 5

Solution:

$$\begin{aligned} E(X^2) &= (-4)^2 \times Pr[X = -4] + (0)^2 \times Pr[X = 0] + (4)^2 \times Pr[X = 4] \\ &= 16 \times \frac{1}{2} + 0 \times \frac{1}{4} + 16 \times \frac{1}{4} = 8 + 0 + 4 = 12 \\ \Rightarrow V(X) &= E(X^2) - (\mu)^2 = 12 - (-1)^2 = 12 - 1 = 11 \end{aligned}$$

A22. X is a random variable with mean $\mu = 2$ and standard deviation $\sigma = 3$. Find $E(X^2)$.

A: 5	B: 7	C: 13	D: -1	E: 11

Solution: We know that $\mu = 2$ and $V(X) = \sigma^2 = 3^2 = 9$, and that $V(X) = E(X^2) - (\mu)^2$, so we see that

$$E(X^2) = V(X) + (\mu)^2 = 9 + (2)^2 = 9 + 4 = 13$$

A23. X is a random variable with variance $V(X) = 4$. Find $V(-2X + 3)$.

A: -8	B: -5	C: 11	D: 16	E: 8

Solution: We use the formula $V(aX + b) = a^2V(X)$, with $a = -2$ and $b = 3$ to get

$$V(-2X + 3) = (-2)^2V(X) = 4(4) = 16$$

A24. X is a random variable with variance $V(X) = 4$. Find $\sigma(-2X + 17)$.

A: 9	B: 3	C: 81	D: 16	E: 4

Solution: We know that $\sigma(X) = \sqrt{V(X)} = \sqrt{4} = 2$, and we need the formula $\sigma(aX + b) = |a|\sigma(X)$ where $a = -2$ and $b = 17$. This gives

$$\sigma(-2X + 17) = |-2|\sigma(X) = 2(2) = 4$$

Alternatively, we could have found $V(-2X + 17)$ and taken the square root:

$$V(-2X + 17) = (-2)^2V(X) = 4(4) = 16 \Rightarrow \sigma(-2X + 17) = \sqrt{V(-2X + 17)} = \sqrt{16} = 4$$

A25. X and Y are independent random variables with $V(X) = 6$ and $V(Y) = 3$. Which of the following **must be true**?

$$\begin{aligned} (i) \quad V(X+Y) &= 9 \\ (ii) \quad V(X-Y) &= 3 \\ (iii) \quad \sigma(X-Y) &= 3 \end{aligned}$$

A: (i) and (ii) only	B: (i) and (iii) only	C: (ii) and (iii) only	D: all of them	E: none of them

Solution: We know that when X and Y are independent random variables, $V(X + Y) = V(X) + V(Y)$, so we have $V(X + Y) = 6 + 3 = 9$ and we see that (i) is true. And since X and Y are independent, then so are X and $-Y$, but that gives

$$V(X - Y) = V(X + (-Y)) = V(X) + V(-Y) = V(X) + (-1)^2V(Y) = V(X) + V(Y) = 9$$

and so (ii) is false. However, this gives $\sigma(X - Y) = \sqrt{V(X - Y)} = \sqrt{9} = 3$, so (iii) is true.

A26. X and Y are independent random variables with $E(X) = -4$ and $E(Y) = 5$. Find $E(XY + 10)$.

A: -20	B: -10	C: 20	D: 30	E: Cannot be determined.

Solution: We know that when X and Y are independent random variables, $E(XY) = E(X) \times E(Y)$. And we know that for any random variable W, $E(aW + b) = a \times E(W) + b$. So we see that

$$E(XY + 10) = E(XY) + 10 = E(X) \times E(Y) + 10 = (-4)(5) + 10 = -20 + 10 = -10$$

A27. X and Y are independent random variables with $V(X) = 1$ and $V(Y) = 4$. Find $\sigma(3X - 2Y)$.

A: -5	B: -7	C: 25	D: 5	E: $\sqrt{3} + \sqrt{8}$

Solution: Since X and Y are independent then so are $3X$ and $-2Y$, so we see that

$$V(3X - 2Y) = V(3X + (-2Y)) = V(3X) + V(-2Y) = 3^2V(X) + (-2)^2V(Y) = 9(1) + 4(4) = 9 + 16 = 25$$

Therefore $\sigma(3X - 2Y) = \sqrt{V(3X - 2Y)} = \sqrt{25} = 5$.

A28. Let X be a continuous random variable with probability density function $f(x) = kx$ for $0 \le x \le 2$ and $f(x) = 0$ otherwise. Find $Pr[X \le k]$.

A: $\frac{1}{4}$	B: $\frac{1}{16}$	C: $\frac{1}{2}$	D: 0	E: 1

Solution: The graph of the density function runs along the x-axis everywhere to the left of $x = 0$ and to the right of $x = 2$. Between these values, it is the line segment joining $(0, f(0)) = (0, 0)$ and $(2, f(2)) = (2, 2k)$. So the region which lies below $y = f(x)$ and above the x-axis is a triangle with base 2 and height $2k$. The area of this region must be 1, so we have

$$\frac{1}{2} \times 2 \times 2k = 1 \Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$$

That is, we have $f(x) = \frac{x}{2}$ for $0 \le x \le 2$. And since $k = \frac{1}{2}$, we see that

$$Pr[X \le k] = Pr\left[X \le \frac{1}{2}\right] = Pr[X < 0] + Pr\left[0 \le X \le \frac{1}{2}\right] = Pr\left[0 \le X \le \frac{1}{2}\right]$$

This probability is given by the area of the triangle with base $\frac{1}{2}$ and height $f\left(\frac{1}{2}\right) = \frac{1/2}{2} = \frac{1}{4}$, so we get

$$Pr[X \le k] = Pr\left[0 \le X \le \frac{1}{2}\right] = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{16}$$

A29. The discrete random variable X has been approximated by a continuous random variable Y. The possible values of X are -12, -8, -4, 0, 4, 8 and 12. Which one of the following gives an approximation of $Pr[X = 0]$?

A: $Pr[-2 < Y < 2]$	B: $Pr[-4 < Y < 4]$	C: $Pr[Y = 0]$	D: $Pr[-0.5 < Y < 0.5]$	E: $Pr[0 < Y < 4]$

Solution: We see that the possible values of X are evenly spaced $w = 4$ units apart, so we need to apply a continuity correction of $\frac{w}{2} = \frac{4}{2} = 2$. Since Y is a good approximation for X (which it must be, or else X could not be approximated by Y), we know that

$$Pr[X = 0] = Pr\left[0 - \frac{w}{2} < Y < 0 + \frac{w}{2}\right] = Pr[0 - 2 < Y < 0 + 2] = Pr[-2 < Y < 2]$$

In the following questions, Z is the standard normal random variable. The Cumulative Distribution Table for Z is given at the back of the exam and may be used whenever required in the following questions.

A30. Find $Pr[Z > 1.57]$.

A: .9418	B: .9332	C: .0582	D: .0668	E: .3920

Solution: The Z-table gives values of $Pr[Z < k]$, so we need that form before we can use the table. We use the complement of the given event (remembering that Z is continuous, so $Pr[Z \le k] = Pr[Z < k]$):

$$Pr[Z > 1.57] = 1 - Pr[Z < 1.57] = 1 - .9418 = .0582$$

(Of course we found $Pr[Z < 1.57] = .9418$ by looking in the 1.5 row and .07 column of the table).

A31. Find $Pr[Z > -2.43]$.

A: .9925	B: .6950	C: .5150	D: .4850	E: .0075

Solution: The Z-table only gives values of $Pr[Z < k]$ for non-negative values of k. When k is negative, we use the symmetry of the Z distribution. That is, the distribution of the standard normal random variable Z is symmetric about its mean, 0, so the probability that Z is greater than some negative value k is the same as the probability that Z is less than a value the same distance away from 0 in the positive direction, which is $-k$ (or you may think of it as $|k|$). Therefore we have

$$Pr[Z > -2.43] = Pr[Z < 2.43] = .9925$$

A32. Find k such that $Pr[k < Z < 0] = .3907$.

A: .1093	B: .8907	C: 1.23	D: −2.31	E: −1.23

Solution: We know that $Pr[k < Z < 0] = Pr[Z < 0] - Pr[Z < k]$. But of course, here k must be negative, since $k < Z < 0$. So we use symmetry to restate $Pr[Z < k]$ as $Pr[Z > -k]$ and then use the complement.

$$\begin{aligned} Pr[k < Z < 0] &= Pr[Z < 0] - Pr[Z < k] &&= Pr[Z < 0] - Pr[Z > -k] \\ &= Pr[Z < 0] - (1 - Pr[Z < -k]) &&= Pr[Z < 0] + Pr[Z < -k] - 1 \end{aligned}$$

And of course we know that (or look in the table to see that, if you forgot) $Pr[Z < 0] = .5$, which gives

$$Pr[k < Z < 0] = .5 + Pr[Z < -k] - 1 = Pr[Z < -k] - .5$$

And since $Pr[k < Z < 0] = .3907$, we must have

$$Pr[Z < -k] - .5 = .3907 \Rightarrow Pr[Z < -k] = .3907 + .5 = .8907$$

From the Z-table, we see that $.8907 = Pr[Z < 1.23]$, so $Pr[Z < -k] = Pr[Z < 1.23]$ and we must have $-k = 1.23$, which gives $k = -1.23$.

A33. If X is a normal random variable with $\mu = \sigma = 2a$, find $Pr[X < 5a]$.

A: .6915	B: .8413	C: .9332	D: .9987	E: Cannot be determined.

Solution: We standardize, by subtracting the mean and then dividing by the standard deviation. We get

$$Pr[X < 5a] = Pr\left[Z < \frac{5a - \mu}{\sigma}\right] = Pr\left[Z < \frac{5a - 2a}{2a}\right] = Pr\left[Z < \frac{3a}{2a}\right] = Pr\left[Z < \frac{3}{2}\right] = Pr[Z < 1.50] = .9332$$

A34. X is a normal random variable with mean $\mu = 3$ and variance $\sigma^2 = 4$. Find $Pr[2 < X < 4]$.

A: .8413	B: .3830	C: .1587	D: .6170	E: .5279

Solution: Since the variance is $\sigma^2 = 4$, then the standard deviation is $\sigma = \sqrt{4} = 2$. So we get

$$\begin{aligned} Pr[2 < X < 4] &= Pr\left[\frac{2-3}{2} < Z < \frac{4-3}{2}\right] &&= Pr\left[-\frac{1}{2} < Z < \frac{1}{2}\right] \\ &= Pr[Z < 0.5] - Pr[Z < -0.5] &&= Pr[Z < 0.5] - Pr[Z > 0.5] \\ &= Pr[Z < 0.5] - (1 - Pr[Z < 0.5]) &&= Pr[Z < 0.50] + Pr[Z < 0.50] - 1 \\ &= 2 \times Pr[Z < 0.50] - 1 &&= 2(.6915) - 1 \\ &= .3830 \end{aligned}$$

A35. X is a normal random variable with mean $\mu = -10$ and standard deviation $\sigma = 5$. Find k such that $Pr[X < k] = .9772$.

A: 0	B: 20	C: 5	D: 15	E: −20

Solution:

$$Pr[X < k] = Pr\left[Z < \frac{k - (-10)}{5}\right] = Pr\left[Z < \frac{k+10}{5}\right]$$

And from the Z-table we see that $.9772 = Pr[Z < 2.00]$. Therefore we have

$$Pr\left[Z < \frac{k+10}{5}\right] = Pr[Z < 2.00] \Rightarrow \frac{k+10}{5} = 2.00 \Rightarrow k + 10 = 5(2) = 10 \Rightarrow k = 0$$

PART B (15 marks)

B1. [2 marks] How many different arrangements of all of the letters in the word BOOKKEEPER do not have the B at the beginning?

Solution: The word BOOKKEEPER contains one B, 2 Os, 2 Ks, 3 Es, one P and one R, for a total of 10 letters. The number of arrangements of all 10 letters is given by the multinomial coefficient $\binom{10}{1\ 2\ 2\ 3\ 1\ 1}$. The number of those arrangements which *do* have the B at the beginning is equal to the number of ways of arranging the other 9 letters, after the B. And we don't want to include these, so we subtract the number of these from the total number of arrangements of the letters. Therefore we see that the number of different arrangements of all 10 letters which do not have the B at the beginning is

$$\binom{10}{1\ 2\ 2\ 3\ 1\ 1} - \binom{9}{2\ 2\ 3\ 1\ 1} \text{ which could also be expressed as } \frac{10!}{2!2!3!} - \frac{9!}{2!2!3!}$$

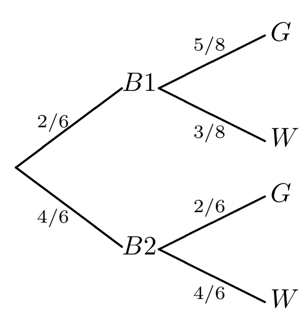
Alternatively, we can count this directly by deciding which 1 of the 9 *other* positions the B is in, and then arranging the remaining 9 letters into the remaining positions. This gives the number of arrangements in which the B is not at the beginning as

$$\binom{9}{1}\binom{9}{2\ 2\ 3\ 1\ 1} = 9 \times \frac{9!}{2!2!3!}$$

B2. [3 marks] There are two boxes. *Box 1* contains 5 green balls and 3 white balls. *Box 2* contains 2 green balls and 4 white balls. A fair die is rolled and if the number rolled is less than 3, *Box 1* is chosen, but if the number rolled is 3 or more, *Box 2* is chosen. One ball is selected at random from the chosen box.

(a) Draw a probability tree to model this stochastic process. Be sure to label the branches and put the appropriate probability on each branch of the tree.

Solution: The first experiment is to choose a box. *Box 1* is chosen (event $B1$) if the die shows a number less than 3, i.e. shows a 1 or a 2, so $Pr[B1] = \frac{2}{6}$. *Box 2* is chosen (event $B2$), when the die shows a 3, 4, 5 or 6, so $Pr[B2] = \frac{4}{6}$. These events and probabilities go on the first level of the tree. The second experiment is to select one ball from the chosen box. The ball will either be green (event G) or white (event W). If *Box 1* was chosen, then the probability of G is $\frac{5}{8}$ and the probability of W is $\frac{3}{8}$. But if *Box 2* was chosen, the probability of G is $\frac{2}{6}$ and the probability of W is $\frac{4}{6}$. Events G and W, and the corresponding conditional probabilities, go on the second level of the tree.



(b) What is the probability that *Box 1* is chosen, or a white ball is drawn, but not both?

Solution: Let E be the event that *Box 1* is chosen, or a white ball is drawn, but not both. Then E occurs when $B1$ occurs and W does not, and also when W occurs and $B1$ does not. To find the probability that E occurs, we sum the path probabilities for all paths on which E occurs, which are the $B1$ then G path and the $B2$ then W path. Therefore we see that

$$Pr[E] = \left(\frac{2}{6} \times \frac{5}{8}\right) + \left(\frac{4}{6} \times \frac{4}{6}\right)$$

B3. [2 marks] The Western football team beats the Waterloo football team 60% of the time when they play. What is the minimum number of games the 2 teams would need to play in order to ensure that the Western team has **more than** a 90% chance of winning at least one of the games?

Solution: Since Western wins 60% of the games played between the two teams, then for each game they play, the probability that Western wins is 0.6 and the probability that Western loses (or at least, doesn't win, if there could be ties) is 0.4. The two teams playing n games corresponds to performing n Bernoulli trials in which, if we consider success to be that Western wins, the probability of success is $p = 0.6$ and the probability of failure is $1 - p = 0.4$. Letting E be the event that Western does win at least one game, we need to find the smallest value of n for which $Pr[E] > .9$. Of course, the complement of winning at least one game is to win no games, so using the Bernoulli formula we see that

$$Pr[E] = 1 - Pr[E^c] = 1 - \binom{n}{0}(0.6)^0(0.4)^n = 1 - 1 \times 1 \times (0.4)^n = 1 - (0.4)^n$$

So requiring this probability to be at least .9 we see that

$$Pr[E] > 0.9 \Rightarrow 1 - (0.4)^n > 0.9 \Rightarrow (0.4)^n < 1 - 0.9 \Rightarrow (0.4)^n < 0.1$$

(That is, we need $Pr[E^c] < 0.1$.) We use trial and error to find the smallest value of n for which this is true. This may be easier if we express 0.4 as $\frac{4}{10}$. Of course, $0.4 > 0.1$, so n is bigger than 1. We see that

$$\left(\frac{4}{10}\right)^2 = \frac{16}{100} = .16 > .1$$

so $n = 2$ is not big enough either. But for $n = 3$ we get

$$\left(\frac{4}{10}\right)^3 = \frac{16}{100} \times \frac{4}{10} = \frac{64}{1000} = 0.064 < 0.1$$

Therefore they should play at least 3 games.

B4. [3 marks] $F(x)$ is the cumulative distribution function (cdf) for a discrete random variable X whose possible values are integers. The following values of $F(x)$ are known:

$$F(5) = 0.42, \quad F(6) = 0.45, \quad F(8) = 0.57, \quad F(9) = 0.68, \quad F(10) = 0.81$$

(a) Find $Pr[X = 6]$.

Solution: The cdf of X gives values of $F(x) = Pr[X \leq x]$, so we know that $Pr[X \leq 6] = F(6) = 0.45$. We need to subtract from this $Pr[X < 6]$. And since the possible values of X are integers, then there cannot be any possible value of X that is less than 6 and is more than 5, so having X be less than 6 means having X be at most 5, so we see that

$$Pr[X = 6] = Pr[X \leq 6] - Pr[X < 6] = Pr[X \leq 6] - Pr[X \leq 5] = F(6) - F(5) = 0.45 - 0.42 = 0.03$$

(That is, the next possible value of X down from 6 is 5, so we know that $Pr[X = 6] = F(6) - F(5)$.)

(b) Find $Pr[5 < X < 9]$.

Solution: We need to find the probability that X has a value which is strictly bigger than 5 and is strictly smaller than 9. Having X be strictly smaller than 9 means it must be at most 8 (the next possible value down from 9). And if we take all of the probability that X is at most 8, we have included the probability for some values we don't want: the ones that are *not* bigger than 5, i.e. the ones that are 5 or less. We must subtract off the probability associated with those values. So we see that

$$\begin{aligned} Pr[5 < X < 9] &= Pr[5 < X \leq 8] &&= Pr[X \leq 8] - Pr[(X > 5)^c] \\ &= Pr[X \leq 8] - Pr[X \leq 5] &&= F(8) - F(5) \\ &= 0.57 - 0.42 &&= 0.15 \end{aligned}$$

(c) Find $Pr[X \geq 9]$.

Solution:

$$Pr[X \geq 9] = 1 - Pr[(X \geq 9)^c] = 1 - Pr[X < 9] = 1 - Pr[X \leq 8] = 1 - 0.57 = 0.43$$

B5. [5 marks] At King's University College, for each student who is registered in Math 1228 at the start of the term, there is a 10% chance they will drop the course before the Final Exam. A particular Math 1228 class at King's had 100 students registered at the beginning of the term. Let X be the number of these students still taking the course at the end of the term.

(a) Find the mean, μ, and standard deviation, σ, of X.

Solution: Since each student has probability .1 of dropping the course, then each student has probability .9 of still being in the course at the end of the term. Observing how many of the 100 students registered at the start of the term are still in the course at the end of the term corresponds to performing $n = 100$ Bernoulli trials in which the probability of success (defined as still being in the course) is $p = .9$. And X is counting the number of successes, so we have $X = B(100, .9)$. Therefore the mean and standard deviation of X are

$$\begin{aligned} \mu &= np = 100(.9) = 90 \\ \text{and } \sigma &= \sqrt{n(p)(1-p)} = \sqrt{100(.9)(.1)} = \sqrt{90(.1)} = \sqrt{9} = 3 \end{aligned}$$

(b) Use a normal approximation and the table provided to find the probability that there are *more than* 94 students still taking the course at the end of the term.

Solution: We are asked to find $Pr[X > 94]$. Since $np = 90 > 5$ and also $n(1-p) = 100(.1) = 10 > 5$, then we know that X is approximately normal and so using a normal approximation is justified. To approximate the discrete random variable X by a normal random variable (which of course is continuous), we need to apply a continuity correction. The possible values of X are the integers from 0 to 100, which are evenly spaced 1 unit apart, so the continuity correction we need is (as always for a Binomial random variable) .5. Letting Y be the normal random variable with the same mean and standard deviation as X, we get

$$\begin{aligned} Pr[X > 94] &= 1 - Pr[X \leq 94] \\ &= 1 - Pr[Y \leq 94 + .5] = 1 - Pr[Y \leq 94.5] \\ &= 1 - Pr\left[Z \leq \frac{94.5 - \mu}{\sigma}\right] = 1 - Pr\left[Z \leq \frac{94.5 - 90}{3}\right] \\ &= 1 - Pr\left[Z \leq \frac{4.5}{3}\right] = 1 - Pr[Z \leq 1.5] \\ &= 1 - .9332 = .0668 \end{aligned}$$