

1. Find an equation for the plane that contains the two lines given by parametric equations

$$x = 1 + t, y = -1 - t, z = -1 + 2t$$

and

$$x = -1 + 2t, y = -4 + 3t, z = -3 + 2t.$$

Cross (X) the box for the correct answer:

A $2x + 2y + 2z = 0$

B $-8x - y - z = -15$

C $2x + 3y + 2z = -15$

D $x - y + 5z = 0$

E $-8x + 2y + 5z = -15$ **Correct**

F $-8x + 2y + 5z = 0$

Solution: A normal vector for the plane is given by $\vec{d}_1 \times \vec{d}_2$, where $\vec{d}_1 = (1, -1, 2)$ and $\vec{d}_2 = (2, 3, 2)$ are direction vectors of the lines above. So we obtain

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & 3 & 2 \end{vmatrix} = (-8, 2, 5)$$

Since the plane contains the point $(1, -1, -1)$, we get $\vec{n} \cdot (1, -1, -1) = -15$.

Hence, the desired equation is $-8x + 2y + 5z = -15$.

2. Find an equation of the plane that is parallel to the z -axis and passes through the points $P = (-3, 1, 2)$ and $Q = (1, 2, -2)$.

Cross (X) the box for the correct answer.

A $x + y + z = 0$

B $x - 4y = -7$ **Correct**

C $x - 4y = 3$

D $x + 2y - 2z = 0$

E $-x + y + 2z = 1$

F $-x + y + 2z = 0$

Solution. Such a plane is parallel to the vectors $(1, 2, -2) - (-3, 1, 2) = (4, 1, -4)$ and $(0, 0, 1)$ (as it is parallel to the z -axis). Therefore, a normal vector is

$$\begin{vmatrix} i & j & k \\ 4 & 1 & -4 \\ 0 & 0 & 1 \end{vmatrix} = (1, -4, 0).$$

Since this plane passes through P we have $(-3, 1, 2) \cdot (1, -4, 0) = -7$. Hence, the equation is $1x - 4y = -7$.

3. Find an equation of the plane that passes through the point $(-5, 2, -1)$ and is perpendicular to the line whose parametric equations are

$$x = 7 - 4t, \quad y = 2 + 2t, \quad z = -3 + t.$$

Cross (X) the box for the correct answer:

A $-4x + 2y + z = -23$

B $-4x + 2y + z = 23$ **Correct**

C $2x - 4y + z = -10$

D $2x - 4y + z = 10$

E $7x + y - 3z = -71$

F $-4x + 2y + z = -28$

Solution: A normal vector \vec{n} for the plane will be the direction vector of the given line, which is $(-4, 2, 1)$.

Since the plane contains the point $(-5, 2, -1)$, we get $\vec{n} \cdot (-5, 2, -1) = 23$. Hence, the equation for the plane is $-4x + 2y + z = 23$.

4. Find parametric equations for the line containing points $(-8, 6, 6)$ and $(-2, 2, 3)$.

Cross (X) the box for the correct answer:

A $x = -2 - 6t, y = 2 + 4t, z = 3 + 3t$ **Correct**

B $x = -1 - 6t, y = 3 + 4t, z = 6 + t$

C $x = -2 + 4t, y = 2 - 2t, z = 3$

D $x = -1 - 6t, y = 1 - t, z = 4 + 3t$

E $x = 4 - 6t, y = -2 + 4t, z = 1 + 3t$

F Such a line does not exist

Solution:

A direction vector for such a line is the difference $(-8, 6, 6) - (-2, 2, 3) = (-6, 4, 3)$. There are two lines above with this direction vector, namely A and E. But only A passes through both of the given points, so the correct answer is A.

5. Which two of the following are vector parametric equations of the plane with Cartesian equation $x - 2y + z = 4$?

I. $v = (0, 0, 0) + s(2, 1, 0) + t(0, 1, 2)$

II. $v = (0, 0, 4) + s(-1, 0, 1) + t(2, 1, 0)$

III. $v = (0, -2, 0) + s(-1, 0, 1) + t(0, 1, 2)$

IV. $v = (0, 0, 4) + s(1, 1, 1) + t(1, 0, 1)$

Cross (X) the box for the correct answer:

A I. and II.

B I. and III.

C I. and IV.

D III. and IV.

E II. and III. **Correct**

F II. and IV.

Solution: Since the plane does not contain $(0, 0, 0)$, we can exclude I.

When $v = P + sv_1 + tv_2$ is the vector parametric equation of a plane, the vectors v_1 and v_2 must be perpendicular to any normal vector; in this case it is $(1, -2, 1)$.

Simple checks show that $(-1, 0, 1)$, $(2, 1, 0)$, $(1, 1, 1)$, $(0, 1, 2)$ are perpendicular to $(1, -2, 1)$, but $(1, 0, 1)$ is not. So IV. can not be correct.

6. Which of the vectors below is perpendicular (orthogonal) to both $(1, 2, -1)$ and $(1, 3, 5)$?

Cross (X) the box for the correct answer:

A $(4, 9, 3)$

B $(-13, -6, 1)$

C $(-13, 6, -1)$ **Correct**

D $(1, -6, 13)$

E $(-6, 13, 1)$

F None of the above

Solution: Such a vector will be parallel to

$$(1, 2, -1) \times (1, 3, 5) = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 1 & 3 & 5 \end{vmatrix} = (13, -6, 1).$$

7. Let $A = (2, 4, 1)$, $B = (3, 0, 9)$, and $C = (1, 4, 0)$. Find the angle $\angle BAC$.

Cross (X) the box for the correct answer:

A $\pi/6$

B $\pi/4$

C $\pi/3$

D $\pi/2$

E $3\pi/4$ **Correct**

F $4\pi/3$

Solution: We find $\vec{u} = C - A = (1, 4, 0) - (2, 4, 1) = (-1, 0, -1)$ and $\vec{v} = B - A = (1, -4, 8)$. Let $\theta = \angle BAC$. Then

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} = \frac{(-1, 0, -1) \cdot (1, -4, 8)}{\sqrt{2}\sqrt{81}} = \frac{-9}{9\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Hence, $\theta = 3\pi/4$.

8. Let L be the line passing through the points $P = (-1, 1, 0)$ and $Q = (-3, 4, 2)$. The point of intersection of L with the plane given by the equation $x + y - z = 1$ is:

Cross (X) the box for the correct answer:

A $(1, 0, 0)$

B $(1, -2, -2)$ **Correct**

C $(0, -1, -1)$

D $(-1, 0, -1)$

E $(\frac{1}{2}, \frac{1}{2}, 0)$

F $(0, \frac{1}{2}, -\frac{1}{2})$

Solution:

The line L has the direction vector $(-3, 4, 2) - (-1, 1, 0) = (-2, 3, 2)$. So it is given by the following parametric equations

$$x = -1 + -2t, \quad y = 1 + 3t, \quad z = 0 + 2t. \quad (*)$$

This line intersects the plane $x + y - z = 1$ when

$$(-1 + (-2)t) + (1 + 3t) - (0 + 2t) = 1,$$

which implies $-1 + (-1)t = 0$. Hence, $t = -1$.

Substituting into (*) we obtain the point of intersection $(1, -2, -2)$.

9. If $\vec{u} = (1, 3, -1)$ and $\vec{v} = (1, 1, 1)$, then the orthogonal projection of \vec{u} onto \vec{v} is:

Cross (X) the box for the correct answer:

A $(1, 3, -1)$

B $4(1, 1, 1)$

C $4(1, 3, -1)$

D $-(1, 1, 1)$

E $(1, 1, 1)$ **Correct**

F $-(1, 3, -1)$

Solution:

We obtain

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \frac{(1, 3, -1) \cdot (1, 1, 1)}{1^2 + 1^2 + 1^2} (1, 1, 1) = \frac{3}{3} (1, 1, 1).$$

10. Find the area of the triangle whose vertices are the points $A = (2, -1, 0)$, $B = (0, -1, 1)$, and $C = (1, 0, -2)$.

Cross (X) the box for the correct answer:

A $\sqrt{30}$

B $2\sqrt{30}$

C $\frac{1}{2}\sqrt{30}$ **Correct**

D 15

E $\sqrt{15}$

F 30

Solution:

We find $\vec{u} = B - A = (0, -1, 1) - (2, -1, 0) = (-2, 0, 1)$ and $\vec{v} = C - A = (1, 0, -2) - (2, -1, 0) = (-1, 1, -2)$.

Then

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ -2 & 0 & 1 \\ -1 & 1 & -2 \end{vmatrix} = (-1, -5, -2).$$

The area is

$$\frac{1}{2} \|\vec{u} \times \vec{v}\| = \frac{1}{2} \sqrt{(-1)^2 + (-5)^2 + (-2)^2} = \frac{1}{2} \sqrt{30}.$$

Useful formulas and more space for work

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, & \sin\left(\frac{\pi}{3}\right) &= \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\ \sin\left(\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, & \sin(0) &= \cos\left(\frac{\pi}{2}\right) = 0, & \sin\left(\frac{\pi}{2}\right) &= \cos(0) = 1.\end{aligned}$$