Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 1341C — Diagnostic Test (V1) **Instructor: Mateja Šajna**

Last name: ______________________________

First name: ______________________________

Student number: ____________________ Signature: ____________________

Circle: DGD1 (MRT 250) DGD2 (FSS 1006) DGD3 (STE G0103) DGD4 (TBT 325)

Instructions:

- This is an 80-minute **closed-book** exam; **no notes** are permitted. Only **calculators without a SIN button** are permitted.
- The test consists of 10 **multiple-choice questions**. For each question, **cross the box in front of the correct answer**. Use the rest of the page for your computation, and the back of the page for rough work. Page 12 contains some useful formulas.
- Carefully read each question, and check your work wherever possible. You may ask for clarification.
- Unauthorized electronic devices (such as cellular phones) are not permitted during this test. Such devices must be **turned off completely** and stored out of students' reach (not in pockets). Academic fraud allegations may be filed if a student is found in possession of such a device during the test.

Question	1	2	3	4	5	6	7	8	9	10	Total
Mark											
Max	1	1	1	1	1	1	1	1	1	1	10

1. Find an equation for the plane that contains the two lines given by parametric equations

$$x = 2 + t,\ y = -1 - t,\ z = -1 + 2t$$

and

$$x = -1 + 3t,\ y = -4 + 3t,\ z = -3 + 2t.$$

Cross (X) the box for the correct answer:

- [A] $3x + 4y + 2z = 0$
- [B] $-8x - y - z = -26$
- [C] $2x - y + 6z = 0$
- [D] $-8x + 4y + 6z = -26$ **Correct**
- [E] $-8x + 4y + 6z = 0$
- [F] $3x + 3y + 2z = -26$

Solution: A normal vector for the plane is given by $\vec{d_1} \times \vec{d_2}$, where $\vec{d_1} = (1, -1, 2)$ and $\vec{d_2} = (3, 3, 2)$ are direction vectors of the lines above. So we obtain

$$\vec{n} = \vec{d_1} \times \vec{d_2} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 3 & 3 & 2 \end{vmatrix} = (-8, 4, 6)$$

Since the plane contains the point $(2, -1, -1)$, we get $\vec{n} \cdot (2, -1, -1) = -26$.

Hence, the desired equation is $-8x + 4y + 6z = -26$.

2. Find an equation of the plane that is parallel to the z-axis and passes through the points $P = (-1, 1, 2)$ and $Q = (1, 2, 3)$.

Cross (X) the box for the correct answer:

- [A] $-x + y + 2z = 6$
- [B] $x + y + z = 2$
- [C] $x - 2y = -3$ **Correct**
- [D] $x - 2y = 3$
- [E] $x + 2y + 3z = 2$
- [F] $-x + y + 2z = 0$

Solution. Such a plane is parallel to the vectors $(1, 2, 3) - (-1, 1, 2) = (2, 1, 1)$ and $(0, 0, 1)$ (as it is parallel to the z-axis). Therefore, a normal vector is

$$\begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (1, -2, 0).$$

Since this plane passes through P we have $(-1, 1, 2) \cdot (1, -2, 0) = -3$, Hence, the equation is $1x - 2y = -3$.

3. Find an equation of the plane that passes through the point $(-5, 1, 8)$ and is perpendicular to the line whose parametric equations are

$$x = 7 - 4t,\ y = 2 + 2t,\ z = -3 + t.$$

Cross (X) the box for the correct answer:

- [A] $-4x + 2y + z = -30$
- [B] $2x - 4y + z = -10$
- [C] $2x - 4y + z = 10$
- [D] $-4x + 2y + z = 30$ **Correct**
- [E] $7x + y - 3z = -71$
- [F] $-4x + 2y + z = -28$

Solution: A normal vector $\vec{n}$ for the plane will be the direction vector of the given line, which is $(-4, 2, 1)$.

Since the plane contains the point $(-5, 1, 8)$, we get $\vec{n} \cdot (-5, 1, 8) = 30$. Hence, the equation for the plane is $-4x + 2y + z = 30$.

4. Find parametric equations for the line containing points $(-2, 2, 3)$ and $(4, -2, 0)$.

Cross (X) the box for the correct answer:

- [A] $x = -1 - 6t,\ y = 3 + 4t,\ z = 6 + t$
- [B] $x = -2 - 6t,\ y = 2 + 4t,\ z = 3 + 3t$ **Correct**
- [C] $x = -2 + 4t,\ y = 2 - 2t,\ z = 3$
- [D] $x = -1 - 6t,\ y = 1 - t,\ z = 4 + 3t$
- [E] $x = 4 - 6t,\ y = -2 + 4t,\ z = 1 + 3t$
- [F] Such a line does not exists.

Solution:

A direction vector for such a line is the difference $(-2, 2, 3) - (4, -2, 0) = (-6, 4, 3)$. There are two lines above with this direction vector, namely [B] and [E]. But only [B] passes through both of the given points, so the correct answer is [B].

5. Which two of the following are vector parametric equations of the plane with Cartesian equation $x - 2y + z = 4$?

I. $v = (0,0,0) + s(0,1,2) + t(2,1,0)$

II. $v = (0,-2,0) + s(1,0,-1) + t(2,1,0)$

III. $v = (4,0,0) + s(1,1,1) + t(1,0,1)$

IV. $v = (4,0,0) + s(1,0,-1) + t(0,1,2)$

Cross (X) the box for the correct answer:

- A I. and II.
- B I. and III.
- C I. and IV.
- D II. and III.
- E III. and IV.
- F II. and IV. **Correct**

Solution: Since the plane does not contain $(0,0,0)$, we can exclude I.

When $v = P + sv_1 + tv_2$ is the vector parametric equation of a plane, the vectors v_1 and v_2 must be perpendicular to any normal vector; in this case it is $(1,-2,1)$.

Simple checks show that $(1,0,-1)$, $(2,1,0)$, $(1,1,1)$, $(0,1,2)$ are perpendicular to $(1,-2,1)$, but $(1,0,1)$ is not. So III. cannot be correct.

6. Which of the vectors below is perpendicular (orthogonal) to both $(2, 1, -1)$ and $(1, 1, 5)$?

Cross (X) the box for the correct answer:

- [A] None of the vectors below **Correct**
- [B] $(6, 11, 1)$
- [C] $(11, 6, 1)$
- [D] $(9, 4, 3)$
- [E] $(-6, 11, 1)$
- [F] $(4, 9, 3)$

Solution: Such a vector will be parallel to

$$(2, 1, -1) \times (1, 1, 5) = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & 1 & 5 \end{vmatrix} = (6, -11, 1).$$

So none of the vectors above is parallel to $(6, -11, 1)$.

7. Let $A = (2, 4, 1)$, $B = (1, 4, 0)$, and $C = (3, 0, 9)$. Find the angle $\angle BAC$.

Cross (X) the box for the correct answer:

- A $\pi/6$
- B $\pi/4$
- C $3\pi/4$ **Correct**
- D $\pi/3$
- E $\pi/2$
- F $4\pi/3$

Solution: We find $\vec{u} = B - A = (1, 4, 0) - (2, 4, 1) = (-1, 0, -1)$ and $\vec{v} = C - A = (1, -4, 8)$. Let $\theta = \angle BAC$. Then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||} = \frac{(-1, 0, -1)\cdot(1, -4, 8)}{\sqrt{2}\sqrt{81}} = \frac{-9}{9\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Hence, $\theta = 3\pi/4$.

8. Let L be the line passing through the points $P = (1, 1, 0)$ and $Q = (3, 5, 2)$. The point of intersection of L with the plane given by the equation $x + y - z = 1$ is:

Cross (X) the box for the correct answer:

- [A] $(1, 0, 0)$
- [B] $(0, -1, -1)$
- [C] $(-1, 0, -1)$
- [D] $(\frac{1}{2}, \frac{1}{2}, 0)$
- [E] $(0, \frac{1}{2}, -\frac{1}{2})$
- [F] $(\frac{1}{2}, 0, -\frac{1}{2})$ **Correct**

Solution:

The line L has the direction vector $(3, 5, 2) - (1, 1, 0) = (2, 4, 2)$. So it is given by the following parametric equations

$$x = 1 + 2t,\ y = 1 + 4t,\ z = 0 + 2t. \tag{*}$$

This line intersects the plane $x + y - z = 1$ when

$$\big(1 + 2t\big) + \big(1 + 4t\big) - \big(0 + 2t\big) = 1,$$

which implies $1 + 4t = 0$. Hence, $t = -0.25$.

Substituting into (*) we obtain the point of intersection $(0.5, 0, -0.5)$.

9. If $\vec{u} = (2, -1, -1)$ and $\vec{v} = (1, 0, 1)$, then the orthogonal projection of $\vec{u}$ onto $\vec{v}$ is:

Cross (X) the box for the correct answer:

- A $\frac{1}{2}(1, 0, 1)$ **Correct**
- B $2(1, 0, 1)$
- C $\frac{1}{2}(2, -1, -1)$
- D $2(2, -1, -1)$
- E $-(1, 0, 1)$
- F $-(2, -1, -1)$

Solution:

We obtain

$$proj_{\vec{v}}\,\vec{u} = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\,\vec{v} = \frac{(2, -1, -1) \cdot (1, 0, 1)}{1^2 + 0^2 + 1^2}(1, 0, 1) = \tfrac{1}{2}(1, 0, 1).$$

10. Find the area of the triangle whose vertices are the points $A = (1, 1, 0)$, $B = (0, -1, 1)$, and $C = (1, 0, 2)$.

Cross (X) the box for the correct answer:

- [A] 7
- [B] $\frac{1}{2}\sqrt{14}$ **Correct**
- [C] 14
- [D] $\sqrt{14}$
- [E] $2\sqrt{14}$
- [F] $\sqrt{7}$

Solution:

We find $\vec{u} = B - A = (0, -1, 1) - (1, 1, 0) = (-1, -2, 1)$ and $\vec{v} = C - A = (1, 0, 2) - (1, 1, 0) = (0, -1, 2)$.

Then

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ -1 & -2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = (-3, 2, 1).$$

The area is

$$\frac{1}{2}||\vec{u} \times \vec{v}|| = \frac{1}{2}\sqrt{(-3)^2 + (2)^2 + (1)^2} = \frac{1}{2}\sqrt{14}.$$

Useful formulas and more space for work

$$\sin(\tfrac{\pi}{6}) = \cos(\tfrac{\pi}{3}) = \tfrac{1}{2}, \quad \sin(\tfrac{\pi}{3}) = \cos(\tfrac{\pi}{6}) = \tfrac{\sqrt{3}}{2}$$
$$\sin(\tfrac{\pi}{4}) = \cos(\tfrac{\pi}{4}) = \tfrac{\sqrt{2}}{2}, \quad \sin(0) = \cos(\tfrac{\pi}{2}) = 0, \quad \sin(\tfrac{\pi}{2}) = \cos(0) = 1.$$