1. True or False:

[1 mark] (i) For any two vectors $\mathbf{u}, \mathbf{v} \in V_3$, we have $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$.

A: YES B: NO

$\vec{i} \cdot \vec{j} = 0$ $|\vec{i}| = 1$ and $|\vec{j}| = 1$

[1 mark] (ii) For any two vectors $\mathbf{u}, \mathbf{v} \in V_3$, we have $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

A: YES B: NO

That's one of the properties of the dot product.

[1 mark] (iii) For any two vectors $\mathbf{u}, \mathbf{v} \in V_3$, we have $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$.

A: YES B: NO

because $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$

[1 mark] (iv) For any two vectors $\mathbf{u}, \mathbf{v} \in V_3$, we have $(\mathbf{u} + \mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v}$.

A: YES B: NO

$(\vec{u} + \vec{v}) \times \vec{v} = \vec{u} \times \vec{v} + \vec{v} \times \vec{v} = \vec{u} \times \vec{v} + \vec{0} = \vec{u} \times \vec{v}$

[1 mark] (v) Given $\mathbf{u}, \mathbf{v} \in V_3$, if $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

A: YES B: NO

Assume $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$
Since $\vec{u} \cdot \vec{v} = 0$, it follows that $\vec{u}$ and $\vec{v}$ are orthogonal.
Then $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \frac{\pi}{2} = |\vec{u}||\vec{v}| \neq 0$ contradiction

[1 mark] (vi) The vector $\langle 6, -2, 14 \rangle$ is parallel to the plane $3x - y + 7z = 1$.

A: YES B: NO

this vector is perpendicular to this plane

$\langle 6, -2, 14 \rangle = 2 \langle 3, -1, 7 \rangle$

Note: the first five questions are from p. 882 of the textbook.

[3 marks] 2. If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the standard basis vectors in V_3, then the vector $2\mathbf{i} \times (3\mathbf{j} - 4\mathbf{k})$ is equal to

A: $24\mathbf{i} \cdot \mathbf{j} \cdot \mathbf{k}$	B: $-2\mathbf{k}$	C: $8\mathbf{j} + 6\mathbf{k}$	D: $6\mathbf{k} - 8\mathbf{j}$	E: $\mathbf{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 0 \\ 0 & 3 & -4 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(-8-0) + \vec{k}(6-0) = 8\vec{j} + 6\vec{k}$$

[3 marks] 3. Given points $A(0, -2, 0)$, $B(1, 0, 0)$, and $C(0, 0, 1)$, the area of the triangle ABC is equal to

A: 4	B: $\frac{3\sqrt{2}}{2}$	C: 3	D: $\frac{3}{2}$	E: 0

$$\frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{4+1+4} = \frac{3}{2}$$

$\vec{AB} = \langle 1, 2, 0 \rangle$

$\vec{AC} = \langle 0, 2, 1 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{vmatrix} = \vec{i}(2-0) - \vec{j}(1-0) + \vec{k}(2-0) = \langle 2, -1, 2 \rangle$$

[3 marks] 4. If $\mathbf{u} = \langle 3, 0, 4\rangle$, $\mathbf{v}$ lies in the xy-plane, $|\mathbf{v}| = 5$, and $\text{comp}_{\mathbf{u}}\mathbf{v} = 3$, then $\mathbf{v}$ is

A: $\langle 5, 0, 0\rangle$	B: $\langle 9, 0, 12\rangle$	C: $\langle \frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}, 0\rangle$	D: $\langle 3, 4, 0\rangle$	E: there is no such vector

The length of $\langle 5,0,0\rangle$ is 5, and its third component is zero.

$$\text{comp}_{\vec{u}}\langle 5,0,0\rangle = \frac{\vec{u}\cdot\langle 5,0,0\rangle}{|\vec{u}|} = \frac{15+0+0}{\sqrt{9+0+16}} = 3$$

$2x+y-2z=-3$

[3 marks] 5. The distance between the planes $2x + y - 2z = 0$ and $4z - 2y - 4x = 6$ is

A: 1	B: 2	C: 3	D: 6	E: 0 (the planes intersect)

The planes are parallel (have the same normal vector).

The distance between them is equal to

the distance from $(0,0,0)$ (a point on the first plane)

to the second plane

which is $\dfrac{|0+3|}{\sqrt{4+1+4}} = 1$

[6 marks] 6. Find an equation of the plane that contains the line $x - 1 = 2 - y = \frac{4-z}{3}$ and is parallel to the plane $5x + 2y + z = 2016$.

Parametric equations

$$\begin{cases} x = 1+t \\ y = 2-t \\ z = 4-3t \end{cases}$$

First, we observe: $\langle 1,-1,-3\rangle \cdot \langle 5,2,1\rangle = 5-2-3=0$

The direction vector of the line is orthogonal to the normal vector of the plane. Therefore the line is parallel to the plane and there is (another) plane that contains this line and is parallel to $5x+2y+z=2016$.

Now let's find an equation of this plane.
It is parallel to $5x+2y+z=2016$, therefore we can take $\langle 5,2,1\rangle$ as a normal vector, and an equation is

$$5(x-x_0) + 2(y-y_0) + 1(z-z_0) = 0.$$

To find a point (x_0, y_0, z_0) on the plane, we'll choose a point on the line, e.g. the point where $t=0$

$x_0 = 1+0$
$y_0 = 2-0$
$z_0 = 4-3(0)$
$(x_0, y_0, z_0) = (1,2,4)$

$$5(x-1) + 2(y-2) + \cancel{1}(z-4) = 0$$

$$\boxed{5x+2y+z-13=0}$$

P L

[6 marks] 7. Find an equation of the plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$, and is perpendicular to the plane $x + y - 2z = 1$.

Normal vectors: $\vec{n}_1 = \langle 1, 0, -1\rangle$ $\vec{n}_2 = \langle 0, 1, 2\rangle$ $\vec{n}_3 = \langle 1, 1, -2\rangle$ parallel to P, since it's perpendicular to the normal vectors of P

$\vec{n}_1 \times \vec{n}_2$ is a direction vector of the line L

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \vec{i}\,(0-(-1)) - \vec{j}\,(2-0) + \vec{k}\,(1-0) = \langle 1, -2, 1\rangle$$

The vector

$\langle 1, -2, 1\rangle \times \vec{n}_3$ is a normal vector for the plane P.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = \vec{i}\,(4-1) - \vec{j}\,(-2-1) + \vec{k}\,(1-(-2)) = \langle 3, 3, 3\rangle$$

We can take $\langle 1, 1, 1\rangle$ instead as a normal vector.

An equation of P is

$$1(x - x_0) + 1(y - y_0) + 1(z - z_0) = 0$$

As (x_0, y_0, z_0) we can take any point on L.

For example, if we try setting $z_0 = 0$, then $x - 0 = 1$

and $y + 2(0) = 3$

so $x_0 = 1$ $y_0 = 3$

So the point $(1, 3, 0)$ belongs to L.

(and hence is on P).

The equation is $x - 1 + y - 3 + z - 0 = 0$

$$\boxed{x + y + z = 4}$$

Note: this problem is #40 p.872 from the textbook

[6 marks] 8. If $\mathbf{w} = |\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u}$, where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all non-zero vectors, show that $\mathbf{w}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

Since $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$, all three vectors lie in the same plane.

Let Θ_1 be the angle between $\vec{w}$ and $\vec{u}$, and let Θ_2 be the angle between $\vec{w}$ and $\vec{v}$.

Let's show that $\cos\Theta_1 = \cos\Theta_2$ and the conclusion will follow.

$$\cos\Theta_1 = \frac{\vec{u}\cdot\vec{w}}{|\vec{u}||\vec{w}|} \qquad \cos\Theta_2 = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$$

It suffices to show that $\dfrac{\vec{u}\cdot\vec{w}}{|\vec{u}|} = \dfrac{\vec{v}\cdot\vec{w}}{|\vec{v}|}$.

$$\frac{\vec{u}\cdot\vec{w}}{|\vec{u}|} = \frac{1}{|\vec{u}|}\vec{u}\cdot(|\vec{u}|\vec{v} + |\vec{v}|\vec{u}) = \frac{1}{|\vec{u}|}|\vec{u}|\vec{u}\cdot\vec{v} + \frac{1}{|\vec{u}|}|\vec{v}|\underbrace{\vec{u}\cdot\vec{u}}_{|\vec{u}|^2} = \vec{u}\cdot\vec{v} + |\vec{v}||\vec{u}|$$

$$\frac{\vec{v}\cdot\vec{w}}{|\vec{v}|} = \frac{1}{|\vec{v}|}\vec{v}\cdot(|\vec{u}|\vec{v} + |\vec{v}|\vec{u}) = \frac{1}{|\vec{v}|}|\vec{u}|\underbrace{\vec{v}\cdot\vec{v}}_{|\vec{v}|^2} + \frac{1}{|\vec{v}|}|\vec{v}|\vec{v}\cdot\vec{u} = |\vec{u}||\vec{v}| + \vec{u}\cdot\vec{v}$$

We conclude that $\dfrac{\vec{u}\cdot\vec{w}}{|\vec{u}|} = \dfrac{\vec{v}\cdot\vec{w}}{|\vec{v}|}$

and the desired statement follows.

Note: this problem is #58 p. 854 from the textbook