SUGGESTED SOLUTIONS

Linear Algebra Test 1

Course: MATH 1104B - Linear Algebra for Engineering or Science
Time: September 28, 2016 2:35pm-3:25pm (50 minutes)
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Tutorial group: Bx

Only writing tools and non-programmable calculators are allowed. Write your solutions directly on the test papers. This test has seven questions. The first three are multiple choice and you don't need to explain your answers, just circle one of the options. In the second part you need to show all your calculations and state your answers clearly. Have your campus card ready at your desk. Good luck!

Multiple choice part

1. A linear system has the unique solution $(x_1, x_2, x_3) = (5, 0, 7)$. Which of the following augmented matrices is the reduced row echelon form of the system?

   (a)
   $$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

   (b)
   $$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

   (c)
   $$\begin{bmatrix} 5 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 7 & 1 \end{bmatrix}$$

2. For what value of c are the vectors $\{\begin{bmatrix} 2 \\ c \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\}$ linearly dependent?

   (a) $c = 9$
   (b) $c = -6$
   (c) $c = \frac{2}{3}$
   (d) $c = -\frac{3}{2}$

3. Which of the following vectors is *not* a linear combination of $(1, 0, -1)$ and $(1, 1, 1)$?

   (a) $(0, 0, 0)$
   (b) $(1, 1, 1)$
   (c) $(1, 2, 3)$
   (d) $(1, 2, 1)$

Problem part

4. Solve the linear system $\begin{cases} 2x + 6y + 6z = 4 \\ -x + 3y - 3z = -1 \\ 3x + 5y + 2z = 3 \end{cases}$ $\longleftrightarrow \left[\begin{array}{ccc|c} 2 & 6 & 6 & 4 \\ -1 & 3 & -3 & -1 \\ 3 & 5 & 2 & 3 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 0 & 12 & 0 & 2 \\ -1 & 3 & -3 & -1 \\ 0 & 14 & -7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & \frac{1}{6} \\ -1 & 3 & -3 & -1 \\ 0 & 1 & -\frac{1}{2} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & \frac{1}{6} \\ -1 & 0 & -3 & -\frac{3}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{6} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$$

$\frac{3}{2} - 1 = \frac{1}{2}$

Answer: the system has the unique solution $(x, y, z) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right) = \frac{1}{6}(3, 1, 2)$

5. Consider the following set of vectors: $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   (a) Write the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of the vectors in S.

   (b) Are the vectors of S linearly dependent or independent? Justify your answer!

a) We get the linear system $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 0 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Conclusion: $1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

b) We try to write the zero vector as a lin. comb of S:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

The row echelon form has 3 pivots $\Rightarrow$ there is a unique solution $(0,0,0)$.

Answer: Since the only linear combination equaling zero is the trivial combination, the vectors of S are linearly independent.

6. A linear system in four variables x_1, x_2, x_3, x_4 has the augmented matrix below. Solve the system and express the solutions clearly.

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 1 & 3 \\ 1 & 0 & 2 & 1 & 0 \\ 3 & 1 & 2 & 2 & 3 \end{array}\right] \begin{array}{c} \\ (-2)(-3) \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 0 & 1 & -4 & -1 & 3 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -4 & -1 & 3 \end{array}\right] \begin{array}{c} (-1) \\ \\ \\ \end{array}$$

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$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -4 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longleftrightarrow \boxed{\begin{cases} x_1 = -2x_3 - x_4 \\ x_2 = 3 + 4x_3 + x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}}$$

Answer

$$\left(\text{or vector form} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad s,t \in \mathbb{R} \right)$$

7. Solve the following system for each value of k. State your complete answer clearly.

$$\begin{cases} x + 3y = 2 \\ x + ky = 3 \end{cases} \longleftrightarrow \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 1 & k & 3 \end{array}\right] \begin{array}{c} (-1) \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & (k-3) & 1 \end{array}\right]$$

No sol. if $k = 3$!

Assume $k \neq 3$. Then we continue: (divide by $k-3$):

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{k-3} \end{array}\right] \begin{array}{c} \\ (-3) \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 - \frac{3}{k-3} \\ 0 & 1 & \frac{1}{k-3} \end{array}\right]$$

We note that $2 - \dfrac{3}{k-3} = \dfrac{2(k-3) - 3}{k-3} = \dfrac{2k-9}{k-3}$.

Answer: For $k=3$ there are no solutions.
For $k \neq 3$ there is a unique solution, namely

$$(x,y) = \left(\frac{2k-9}{k-3}, \frac{1}{k-3} \right) = \frac{1}{k-3}(2k-9, 1)$$