1. Let $U = \{(x-y, y, x-y) \mid x,\, y \in \mathbb{R}\}$. Which one of the following statements is **true**?

A U is not a subspace of $\mathbb{R}^3$.
B U is a subspace of $\mathbb{R}^3$ with $\dim U = 3$.
[✓ C] U is a subspace of $\mathbb{R}^3$ and $\{(-1,1,-1),(1,0,1)\}$ is a basis for U.
D U is a line in $\mathbb{R}^3$ with direction vector $(1,1,1)$.
E U is a subspace of $\mathbb{R}^3$ and $\{(1,0,-1),(-1,0,1)\}$ is a basis for U.
F U is a plane in $\mathbb{R}^3$ with normal vector $(-1,2,-1)$.

$$U = \{x(1,0,1) + y(-1,1,-1) \mid x, y \in \mathbb{R}\}$$
$$= \text{span}\{\underset{\uparrow}{(1,0,1)}, \underset{\uparrow}{(-1,1,-1)}\}$$

LI since neither is a scalar multiple of the other

---

2. **Only two** of the following statements are **true**. Find which ones are **true**.

I. $\{3, 5\sin^2 x, 7\cos^2 x\}$ is linearly dependent in the function space $F(\mathbb{R})$.

II. A homogeneous system of linear equations always has a unique solution.

III. If A and B are two 3×3 invertible matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.

IV. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are any non-zero vectors such that $\mathbf{u}-\mathbf{v} = 2\mathbf{u}$, then $\dim(\text{span}\{\mathbf{u}, \mathbf{v}\}) = 1$.

A I and II.
B I and III.
[✓ C] I and IV.
D II and III.
E II and IV.
F III and IV.

I. TRUE since we have a non-trivial dependence equation:

$$-\frac{35}{3}(3) + 7(5\sin^2 x) + 5(7\cos^2 x)$$
$$= -35 + 35\underbrace{(\sin^2 x + \cos^2 x)}_{=1} = 0$$

II. FALSE. The solution may not be unique

e.g. $x+y=0$ has general sol^n $\{(-s, s) \mid s \in \mathbb{R}\}$

III. FALSE $(AB)^{-1} = B^{-1}A^{-1}$

IV. TRUE. $\vec{u} - \vec{v} = 2\vec{u} \iff \vec{v} = -\vec{u}$ $\therefore \vec{v} \in \text{span}\{\vec{u}\}$

and $\therefore \text{span}\{\vec{u}, \vec{v}\} = \text{span}\{\vec{u}\}$ and $\dim(\text{span}\{\vec{u}\}) = 1$ since $\vec{u} \neq \vec{0}$

$\therefore \dim(\text{span}\{\vec{u}, \vec{v}\}) = 1$

3. Let A be an $n \times n$ matrix with $n \geq 2$. Which of the following statements are true?

I. If $\operatorname{rank}(A) = 2$, then there are two free variables in the general solution of the system $A\mathbf{x} = \mathbf{0}$.

II. If $\operatorname{rank}(A) = 1$, then there are $(n-1)$ free variables in the general solution of the system $A\mathbf{x} = \mathbf{0}$.

III. If A is invertible, then the system $A\mathbf{x} = \mathbf{0}$ has a unique solution.

IV. If $\operatorname{rank}(A) < n$, then the system $A\mathbf{x} = \mathbf{0}$ is inconsistent.

A Only I.
B Only II.
C Only I and III.
[✓ D] Only II and III.
E Only I and IV.
F Only II and IV.

I. # parameters = # columns − rank A
$= n - 2 \therefore$ FALSE

II. # parameters = n − rank A
$= n - 1 \quad \therefore$ TRUE

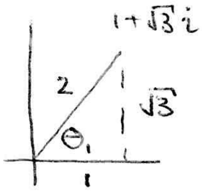
III. $A\vec{x} = \vec{0} \Leftrightarrow A^{-1}A\vec{x} = A^{-1}\vec{0} \Leftrightarrow \vec{x} = \vec{0}$
$\therefore$ TRUE

IV. $A\vec{x} = \vec{0}$ is always consistent since $\vec{x} = \vec{0}$ is a solⁿ
$\therefore$ FALSE

Question 4. What is the polar form of the complex number $\frac{1+\sqrt{3}i}{1-i}$? *(See page 2 for a table of values of trigonometric functions.)*

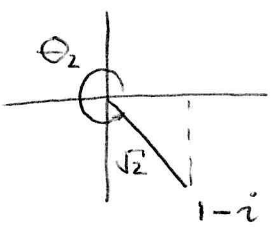
[✓ A] $\sqrt{2}(\cos(7\pi/12) + i\sin(7\pi/12))$
B $\sqrt{2}(\cos(5\pi/12) + i\sin(5\pi/12))$
C $\sqrt{2}(\cos(-\pi/12) + i\sin(-\pi/12))$
D $\sqrt{2}(\cos(\pi/12) + i\sin(\pi/12))$
E $\sqrt{2}(\cos(-5\pi/12) + i\sin(-5\pi/12))$
F $\sqrt{2}(\cos(11\pi/12) + i\sin(11\pi/12))$

Set $z_1 = 1 + \sqrt{3}i$, $z_2 = 1 - i$



$\theta_1 = \frac{\pi}{3}$

$\therefore z_1 = 2e^{\frac{\pi}{3}i}$



$\theta_2 = \frac{7}{4}\pi$

$\therefore z_2 = \sqrt{2}\, e^{\frac{7}{4}\pi i}$

Then

$$\frac{z_1}{z_2} = \frac{2e^{\frac{\pi}{3}i}}{\sqrt{2}\, e^{\frac{7}{4}\pi i}}$$
$$= \frac{2}{\sqrt{2}} e^{(\frac{1}{3} - \frac{7}{4})\pi i}$$
$$= \sqrt{2}\, e^{-\frac{17}{12}\pi i} = \sqrt{2}\, e^{\frac{7}{12}\pi i}$$
$$= \sqrt{2}\left(\cos\left(\tfrac{7\pi}{12}\right) + i\sin\left(\tfrac{7\pi}{12}\right)\right)$$

5. Let $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ and consider the set $U = \{A \in M_{22}(\mathbb{R}) \mid BA = AB\}$. (Recall that $M_{22}(\mathbb{R})$ is the set of 2×2 matrices with entries in $\mathbb{R}$.) Which of the following statements is true?

A U is not a subspace of $M_{22}(\mathbb{R})$.
B U is a subspace of $M_{22}(\mathbb{R})$ and $\dim U = 0$.
C U is a subspace of $M_{22}(\mathbb{R})$ and $\dim U = 1$.
[✓ D] U is a subspace of $M_{22}(\mathbb{R})$ and $\dim U = 2$.
E U is a subspace of $M_{22}(\mathbb{R})$ and $\dim U = 3$.
F U is a subspace of $M_{22}(\mathbb{R})$ and $\dim U = 4$.

Set $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$BA = AB$

$\iff \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

$\iff \begin{bmatrix} a & b \\ -a & -b \end{bmatrix} = \begin{bmatrix} a-b & 0 \\ c-d & 0 \end{bmatrix}$

$\iff b = 0, \quad a = d - c$

$$\therefore U = \left\{A \in M_{22}(\mathbb{R}) \mid A = \begin{bmatrix} d-c & 0 \\ c & d \end{bmatrix}\right\} = \text{span}\left\{\underset{\uparrow}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{\uparrow}{\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}}\right\}$$

LI

---

6. Let $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 6 & -6 & 5 \end{bmatrix}$. What is the third row of A^{-1}?

A $[1\ 2\ 8]$
B $[-1\ -2\ 8]$
[✓ C] $[-8\ -2\ 1]$
D $[8\ 2\ -1]$
E $[2\ 1\ 8]$
F $[2\ 1\ 2]$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 6 & -6 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 11 & -6 & 0 & 1 \end{array}\right]$$

$R_2 + R_1 \to R_2$

$R_3 - 6R_1 \to R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & -8 & -2 & 1 \end{array}\right]$$

$R_3 - 2R_2$

7. For a **non-homogeneous** system of 2019 equations in 1341 variables, answer **Yes** or **No** to the following three questions:

- Can the system be inconsistent? Yes
- Can the system have a unique solution? Yes
- Can the system have infinitely many solutions? Yes

$$\text{e.g.}\quad 2019\updownarrow\overset{\longleftarrow 1341 \longrightarrow}{\left[\begin{array}{ccc|c} & & & \\ 0 & 0 \cdots & 0 & 1 \end{array}\right]}$$

If the system is consistent,

\# parameters

= \# columns − rank A

= 1341 − rank A

= 0 if rank A = 1341

If it's consistent and rank A < 1341

[✓ A] Yes, Yes,Yes
B Yes, Yes, No
C No, Yes, Yes
D Yes, No, Yes
E No, No, Yes
F Yes, No, No

---

8. Let A be an $n \times n$ matrix. Which one of the following statements is equivalent to

"The columns of A are linearly dependent"

A The rows of A are linearly independent.
B rank$(A) = n$.
[✓ C] $\det(A) = 0$.
D The rows of A form a basis for $\mathbb{R}^n$.
E The homogeneous system $A\mathbf{x} = \mathbf{0}$ has a unique solution.
F A is invertible.

The columns of A are LI $\Longleftrightarrow \det(A) \neq 0$

9. Suppose $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\det(A) = 3$. By using the properties of determinants, calculate the determinant of the following matrix:

$$B = \begin{bmatrix} b+5c & e+5f & h+5i \\ 2c & 2f & 2i \\ -2a & -2d & -2g \end{bmatrix}$$

A -64

B -18

[✓ C] -12

D 12

E 18

F 64

$$A \xrightarrow[\text{take transpose}]{\text{no change to det}} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \xrightarrow[\substack{R_1 \leftrightarrow R_3 \\ R_1 \leftrightarrow R_2}]{\times(-1)\times(-1)} \begin{bmatrix} b & e & h \\ c & f & i \\ a & d & g \end{bmatrix}$$

$$\xrightarrow[R_1+5R_2 \to R_1]{\text{no change}} \begin{bmatrix} b+5c & e+5f & h+5i \\ c & f & i \\ a & d & g \end{bmatrix} \xrightarrow[\substack{2R_2 \to R_2 \\ -2R_3 \to R_3}]{\times 2 \times (-2)} \begin{bmatrix} b+5c & e+5f & h+5i \\ 2c & 2f & 2i \\ -2a & -2d & -2g \end{bmatrix} = A$$

$\therefore \det(B) = (-1)^2 (2)(-2) \det(A) = -4 \cdot 3 = -12$

---

10. The set $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\} = \{(3,1,1), (-1,2,1), (-1/2,-2,7/2)\}$ forms an orthogonal basis for $\mathbb{R}^3$. Therefore any vector $\mathbf{v} \in \mathbb{R}^3$ can be written as $\mathbf{v} = a_1\mathbf{u_1} + a_2\mathbf{u_2} + a_3\mathbf{u_3}$, where $a_1, a_2, a_3 \in \mathbb{R}$ are the coordinates of $\mathbf{v}$ with respect to the ordered basis $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$. Find the coordinate a_2 of the vector $\mathbf{v} = (1, 0, -1)$.

A $-2/3$

[✓ B] $-1/3$

C 0

D $1/3$

E $2/3$

F 1

$$a_2 = \frac{\vec{v} \cdot \vec{u}_2}{\|\vec{u}_2\|^2}$$

$$= \frac{(1,0,-1) \cdot (-1,2,1)}{(-1,2,1) \cdot (-1,2,1)}$$

$$= \frac{-1-1}{1+4+1}$$

$$= -\frac{2}{6} = -\frac{1}{3}$$

Question 11. Let

$$U = \operatorname{span}\{\overset{\vec{u}_1}{(-2,1,3,1)}, \overset{\vec{u}_2}{(-5,3,11,7)}, \overset{\vec{u}_3}{(8,-5,-19,-13)}, \overset{\vec{u}_4}{(0,1,7,5)}, \overset{\vec{u}_5}{(-17,5,1,-3)}\} \subseteq \mathbb{R}^4.$$

(a) **[3 points]** Find a basis for U consisting of vectors contained in the above spanning set.

We want a $\underline{\text{subset}}$ of the spanning set.

∴ use column space algorithm.

$$\text{Set } A = \begin{array}{c} \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \\ \begin{array}{ccccc} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 & \vec{u}_5 \end{array} \end{array} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

1 setup

$$\underset{\substack{R_2+2R_1 \to R_2 \\ R_3-3R_1 \to R_3 \\ R_4-R_1 \to R_4}}{\sim} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix} \underset{\substack{R_3-2R_2 \to R_3 \\ R_4-4R_2 \to R_4}}{\sim} \begin{bmatrix} \enclose{circle}{1} & 3 & -5 & 1 & 5 \\ 0 & \enclose{circle}{1} & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \enclose{circle}{-4} & 20 \end{bmatrix}$$

1 row reduction
-0.5 for mistakes

We have pivots in columns 1, 2, & 4

∴ $\{\vec{u}_1, \vec{u}_2, \vec{u}_4\}$ is a basis of U

1/3 if the basis isn't a subset of the given spanning set

Answer: $\{\vec{u}_1, \vec{u}_2, \vec{u}_4\}$ 1 answer (other correct answers are possible)

(b) **[2 points]** Extend the basis you found in (a) to a basis for $\mathbb{R}^4$.

$$\begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 3 & 1 \\ -5 & 3 & 11 & 7 \\ 0 & 1 & 7 & 5 \end{bmatrix} \underset{\substack{-\frac{1}{2}R_1 \to R_1 \\ R_2+5R_1 \to R_2}}{\sim} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{7}{2} & \frac{9}{2} \\ 0 & 1 & 7 & 5 \end{bmatrix}$$

$$\underset{\substack{2R_2 \to R_2 \\ R_3-R_2 \to R_3}}{\sim} \begin{bmatrix} \enclose{circle}{1} & -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & \enclose{circle}{1} & 7 & 9 \\ 0 & 0 & 0 & \enclose{circle}{4} \end{bmatrix}$$

∴ Add $(0,0,1,0)$ to get the missing pivot. 1 justification

Answer: $\{\vec{u}_1, \vec{u}_2, \vec{u}_4, (0,0,1,0)\}$ is a basis of $\mathbb{R}^4$ 1 answer

Question 12. Let $W = \{(w, x, y, z) \in \mathbb{R}^4 \mid w + y - z = 0\} \subseteq \mathbb{R}^4$.

(a) **[2 points]** Find a basis for W.

$$W = \mathrm{Null}\,(\underbrace{[1 \quad 0 \quad 1 \quad -1]}_{A})$$

Sol^n to $A\vec{x} = \vec{0}$:

$w = -s + t$
$x = r$
$y = s$
$z = t$

1 justification

Basic sol^{n}s are $(0, 1, 0, 0)$, $(-1, 0, 1, 0)$, $(1, 0, 0, 1)$
$(r = 1, s = 0, t = 0)$ $(r = 0, s = 1, t = 0)$ $(r = 0, s = 0, t = 1)$

$\therefore \{\underset{\vec{u}_1}{(0, 1, 0, 0)}, \underset{\vec{u}_2}{(-1, 0, 1, 0)}, \underset{\vec{u}_3}{(1, 0, 0, 1)}\}$ is a basis for W

Answer: $\{(0, 1, 0, 0), (-1, 0, 1, 0), (1, 0, 0, 1)\}$ 1 answer

(b) **[3 points]** Use the Gram-Schmidt algorithm to construct an orthogonal basis for W.

$\vec{w}_1 = \vec{u}_1$ 1

$$\vec{w}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1 = (-1, 0, 1, 0) - \frac{(-1, 0, 1, 0) \cdot (0, 1, 0, 0)}{1}(0, 1, 0, 0)$$

$= (-1, 0, 1, 0) - 0$
$= (-1, 0, 1, 0)$

$$\vec{w}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1 - \frac{\vec{u}_3 \cdot \vec{w}_2}{\|\vec{w}_2\|^2}\vec{w}_2$$ 1 formulas

$= (1, 0, 0, 1) - 0 - \frac{-1}{2}(-1, 0, 1, 0)$ -1 for a mistake (since it's easy to check the answer)

$= (\frac{1}{2}, 0, \frac{1}{2}, 1)$

$\therefore \{(0, 1, 0, 0), (-1, 0, 1, 0), (\frac{1}{2}, 0, \frac{1}{2}, 1)\}$ is an orthogonal basis for W

Answer: $\{(0, 1, 0, 0), (-1, 0, 1, 0), (\frac{1}{2}, 0, \frac{1}{2}, 1)\}$ 1 answer

(there are other correct answers)

Question 12: continued...

(c) **[2 points]** Find the best approximation to the vector $\overset{\vec{v}}{(1,1,1,0)}$ by a vector in W.

best approximation $= \text{proj}_W \vec{v}$

$$= \frac{\vec{v}\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1 + \frac{\vec{v}\cdot\vec{w}_2}{\|\vec{w}_2\|^2}\vec{w}_2 + \frac{\vec{v}\cdot\vec{w}_3}{\|\vec{w}_3\|^2}\vec{w}_3$$ 1 formula

$$= \frac{(1,1,1,0)\cdot(0,1,0,0)}{(0,1,0,0)\cdot(0,1,0,0)}(0,1,0,0) + \frac{(1,1,1,0)\cdot(-1,0,1,0)}{(-1,0,1,0)\cdot(-1,0,1,0)}(-1,0,1,0)$$

-0.5 per mistake

$$+ \frac{(1,1,1,0)\cdot(\frac{1}{2},0,\frac{1}{2},1)}{(\frac{1}{2},0,\frac{1}{2},1)\cdot(\frac{1}{2},0,\frac{1}{2},1)}(\frac{1}{2},0,\frac{1}{2},1)$$

$$= (0,1,0,0) + 0(-1,0,1,0) + \frac{1}{\frac{3}{2}}(\frac{1}{2},0,\frac{1}{2},1)$$

$$= (0,1,0,0) + \frac{2}{3}(\frac{1}{2},0,\frac{1}{2},1) = (\frac{1}{3},1,\frac{1}{3},\frac{2}{3})$$

Answer: $\boxed{(\frac{1}{3}, 1, \frac{1}{3}, \frac{2}{3})}$ 1 answer

Question 13. Let $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

(a) [**3 points**] Find the characteristic polynomial of A.

$$p_A(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{bmatrix} 4-\lambda^{+} & 0 & -2 \\ 2^{-} & 5-\lambda & 4 \\ 0^{+} & 0^{-} & 5-\lambda^{+} \end{bmatrix}$$ 1

$$\overset{\text{row3}}{=} (5-\lambda) \begin{vmatrix} 4-\lambda & 0 \\ 2 & 5-\lambda \end{vmatrix}$$ 1

$$= (5-\lambda)^2 (4-\lambda)$$

Answer: $\boxed{p_A(\lambda) = (5-\lambda)^2 (4-\lambda)}$

1 final answer (0.5 for writing the eigenvalues instead)

(b) [**1 point**] Using the characteristic polynomial, explain why the eigenvalues of A are 4 and 5.

These are the eigenvalues because they are the roots of the characteristic polynomial.

1

See next page for the rest of question 13...

Question 13: continued...

(c) [**2 points**] Find a basis for the eigenspace $E_4 = \{\mathbf{v} \in \mathbb{R}^3 \mid A\mathbf{v} = 4\mathbf{v}\}$.

$$E_4 = \text{Null}(A-4I) = \text{Null}\left(\begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_2 \leftrightarrow R_1}{\sim} \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \underset{\substack{\frac{1}{2}R_1 \to R_1 \\ -\frac{1}{2}R_2 \to R_2 \\ R_3 - R_2 \to R_3}}{\sim} \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\uparrow$ t

0.5 setup

0.5 row reducing

sol^n: $x + \frac{1}{2}y + 2z = 0$, $y = t$, $z = 0$ $\Rightarrow$ $x = -\frac{1}{2}t$, $y = t$, $z = 0$

0.5 read solutions

$\therefore \{(-\frac{1}{2}, 1, 0)\}$ is a basis for E_4

Answer: $\boxed{\{(-\frac{1}{2}, 1, 0)\}}$ 0.5

(d) [**2 points**] Find a basis for the eigenspace $E_5 = \{\mathbf{v} \in \mathbb{R}^3 \mid A\mathbf{v} = 5\mathbf{v}\}$.

$$E_5 = \text{Null}(A-5I) = \text{Null}\left(\begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$\begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \underset{\substack{-R_1 \to R_1 \\ R_2 - 2R_1 \to R_2}}{\sim} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\uparrow$ s $\uparrow$ t

same as above

sol^n: $x + 2z = 0$, $y = s$, $z = t$ $\Rightarrow$ $x = -2t$, $y = s$, $z = t$

$\therefore \{(0,1,0), (-2,0,1)\}$ is a basis for E_5

Answer: $\boxed{\{(0,1,0), (-2,0,1)\}}$

(e) [**2 points**] Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Answer: $P = \begin{bmatrix} -\frac{1}{2} & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

1 1

2 marks each = 0.5 answer + 1.5 justification

Question 14. Indicate if each of the following statements is (always) true or is (possibly) false. Put your answers in the boxes indicated and show your work in the spaces below each question.

- If you indicate that the statement is (possibly) false, **you must give an explicit counterexample with numbers.**
- If you indicate that the statement is (always) true, you must give a clear explanation supported with results from class.

a) If A is a 3×4 matrix whose reduced row echelon form (RREF) has a column of zeros, then $\text{rank}(A) < 3$.

0/1.5 for attempting to prove the wrong result

e.g. $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is 3×4

1.5=correct

1 = nearly correct or missing some details/justification

and $\text{rank}(A) = 3$

0.5 = some relevant idea or correct essence

Answer: FALSE 0.5

b) If $A = [0\ 1\ \ -1\ 1]$, then the dimension of $\text{Null}(A)$ is 3.

By the Rank-Nullity Theorem,

$$\dim(\text{Null}(A)) = \#\text{ columns of } A - \text{rank}(A)$$
$$= 4 - 1$$
$$= 3$$

Answer: TRUE 0.5

Question 14: continued...

c) Let z_1 and z_2 be two complex numbers whose imaginary parts are non-zero. Then the imaginary part of their product $z_1 z_2$ is always non-zero.

e.g. If $z_1 = z_2 = i$, then $\text{Im}(z_1) = \text{Im}(z_2) = 1$,

but $z_1 z_2 = i^2 = -1$, so $\text{Im}(z_1 z_2) = 0$.

Answer: FALSE 0.5

d) Suppose $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set of vectors in a vector space V. Then for any non-zero vector $\mathbf{w} \in V$, we have $\dim(\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}) = 3$.

e.g. Take $V = \mathbb{R}^2$ and $\vec{u} = (1,0)$, $\vec{v} = (0,1)$.

Then $\{\vec{u}, \vec{v}\}$ is LI, but if $\vec{w} = \vec{u}$ then $\vec{w} \neq \vec{0}$

and $\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \text{span}\{\vec{u}, \vec{v}\}$,

and so $\dim(\text{span}\{\vec{u}, \vec{v}, \vec{w}\}) = \dim(\text{span}\{\vec{u}, \vec{v}\}) = 2$

ANSWER: FALSE 0.5

Question 15. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-2y \\ 2y-z \\ 2z-x \end{bmatrix}$$

(a) [**2 points**] Find the standard matrix of the linear transformation T.

$$A = [\,T\vec{e}_1 \quad T\vec{e}_2 \quad T\vec{e}_3\,]$$ 1 justification

$$= \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \end{bmatrix}$$

ANSWER: $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \end{bmatrix}$ 1 answer

im(T) = Col(A)

(b)[**3 points**] Find a basis for the image of T, and briefly explain why your set is a basis.

$\Rightarrow$ Use the column space algorithm: 1 for coming up with a worthwhile approach

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \end{bmatrix} \underset{R_3+R_1\to R_3}{\sim} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix} \underset{R_3+R_2\to R_3}{\sim} \begin{bmatrix} \textcircled{1} & -2 & 0 \\ 0 & \textcircled{2} & -1 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

There's a pivot in every column, so columns 1, 2 & 3 form a basis of im(T). 1 for execution

ANSWER: $\{(1,0,-1), (-2,2,0), (0,-1,2)\}$ 1 answer