UNIVERSITY OF MANITOBA

**DATE:** Thursday, October 25th, 2012 **TEST 2**
**TITLE PAGE**
**DEPARTMENT & COURSE NO:** MATH 2132 A01 **TIME:** 1 hour
**EXAMINATION:** Analysis II **EXAMINERS:** O. Gueye/M. Virgilio

SURNAME: (Print) Solution

GIVEN NAME(S): ____________________

STUDENT NUMBER: ____________________

SIGNATURE: ____________________
(I understand that cheating is a serious offense)

**Please indicate your instructor by checking the appropriate box below.**

☐ A01 M. Virgilio

☐ A02 O. Gueye

**INSTRUCTIONS TO STUDENTS:**

This is a 1 hour test. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cells phones or electronic translators permitted.

The value of each question is indicated to the right of the question. The total value of all questions is 100 points.

**Answer all questions on the exam paper** in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

| Questions | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 30 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| 6 | 5 | |
| Total: | 100 | |

1.
Find the open interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(3n)!} x^n.$$

5 Marks

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(2n)!}{(3n)!} \cdot \frac{(3n+3)!}{(2n+2)!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(2n)! \cdot (3n)! \cdot (3n+1)(3n+2)(3n+3)}{(3n)! \cdot (2n)! \cdot (2n+1)(2n+2)}\right|$$

$$= \lim_{n\to\infty} \frac{(3n+1)(3n+2)(3n+3)}{(2n+1)(2n+2)} = \infty$$

The open interval of convergence is $(-\infty, +\infty)$.

2.

Find the sum of the following power series. Give the open interval of convergence.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!}(2x+3)^{2n+2}$ 8 Marks

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!}(2x+3)^{2n+2} = -(2x+3)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}(2x+3)^{2n}$$

$$= -(2x+3)^2 \left(\cos(2x+3) - 1\right)$$

$$= (2x+3)^2 - (2x+3)^2 \cos(2x+3)$$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{(2n)!} \cdot \frac{(2n+2)!}{(-1)^{n+2}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{(-1)^{n+2}} \frac{(2n)!(2n+1)(2n+2)}{(2n)!}\right|$$

$$= \lim_{n\to\infty} (2n+1)(2n+2) = \infty$$

The open interval of convergence is : $(-\infty, \infty)$.

(b) $\sum_{n=2}^{\infty}(n+2)2^n x^{2n+1}$ 22 Marks

If we set $y=x^2$, then $\sum_{n=2}^{\infty}(n+2)2^n x^{2n+1} = x\sum_{n=2}^{\infty}(n+2)2^n x^{2n}$
$= \sqrt{y}\sum_{n=2}^{\infty}(n+2)2^n y^n$

Since $R_y = \lim_{n\to\infty}\left|\frac{(n+2)2^n}{(n+3)2^{n+1}}\right| = \frac{1}{2}\lim_{n\to\infty}\frac{(n+2)}{(n+3)} = \frac{1}{2}$, it follows that $R_x = \sqrt{R_y} = \frac{1}{\sqrt{2}}$. The open interval of convergence is therefore $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

Let $S(x) = \sum_{n=2}^{\infty}(n+2)2^n x^{2n+1}$ then $x^2 S(x) = \sum_{n=2}^{\infty}(n+2)2^n x^{2n+3}$

Term-by-term antidifferentiation of this equation gives

$$\int x^2 S(x)\,dx = \sum_{n=2}^{\infty}\frac{(n+2)}{(2n+4)}2^n x^{2n+4} + C = \sum_{n=2}^{\infty}\frac{2^n}{2}x^{2n+4} + C$$

$$= \frac{x^4}{2}\sum_{n=2}^{\infty}2^n x^{2n} + C = \frac{x^4}{2}\sum_{n=2}^{\infty}(2x^2)^n + C$$

$$= \frac{x^4}{2}\cdot\frac{4x^4}{(1-2x^2)} + C = \frac{2x^8}{1-2x^2} + C$$

Differentiation now gives

$$x^2 S(x) = \frac{16x^7(1-2x^2) + 4x(2x^8)}{(1-2x^2)^2}$$

$$= \frac{16x^7 - 24x^9}{(1-2x^2)^2}, \text{ so}$$

$$S(x) = \frac{16x^5 - 24x^7}{(1-2x^2)^2} \quad \text{for } |x| < \frac{1}{\sqrt{2}}.$$

**3.**
Use the binomial expansion to find the MacLaurin series for

$$f(x) = \sqrt[4]{2-x}.$$

Give the open interval of convergence. Express your final answer in sigma notation.

20 Marks

$$\sqrt[4]{2-x} = \sqrt[4]{2}\,\sqrt[4]{\left(1-\frac{x}{2}\right)} = \sqrt[4]{2}\left(1+\left(-\frac{x}{2}\right)\right)^{\frac{1}{4}}$$

$$= \sqrt[4]{2}\Big[1+\left(\frac{1}{4}\right)\left(-\frac{x}{2}\right)+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}\left(-\frac{x}{2}\right)^2$$

$$+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}\left(-\frac{x}{2}\right)^3+\cdots$$

$$+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\cdots\left(-\frac{(4n-5)}{4}\right)}{n!}\left(-\frac{x}{2}\right)^n+\cdots\Big]$$

$$= \sqrt[4]{2}\Big[1-\frac{x}{4\cdot 2}-\frac{1.3\,x^2}{2!\,4^2\,2^2}-\frac{1.3.7\,x^3}{3!\,4^3\,2^3}$$

$$-\frac{1.3.7.\ldots.(4n-5)}{n!\,4^n\,2^n}x^n+\cdots\Big]$$

$$= \sqrt[4]{2}\left[1-\frac{x}{8}-\sum_{n=2}^{\infty}\frac{1.3.7.\ldots.(4n-5)}{n!\,8^n}x^n\right],$$

valid for $-1 < -\frac{x}{2} < 1$ or $-2 < x < 2$

4.

Find the MacLaurin series of the function

$$f(x) = \frac{1}{x^2 - 3x + 2}.$$

Give the open interval of convergence. Express your final answer in sigma notation.

**20 Marks**

$$f(x) = \frac{1}{x^2-3x+2} = \frac{1}{x-2} - \frac{1}{x-1} = \frac{1}{1-x} - \frac{1}{2-x}$$

$$= \frac{1}{1-x} - \frac{1}{2\left(1-\frac{x}{2}\right)} = \sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n} = \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$$

$$= \sum_{n=0}^{\infty}\left[1 - \frac{1}{2^{n+1}}\right]x^n.$$

For $\sum_{n=0}^{\infty} x^n$, the open interval of convergence is $-1 < x < 1$

For $\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$, the open interval of convergence is $-2 < x < 2$

So, for $f(x)$, the open interval of convergence is

$-1 < x < 1$.

5.

Use term-by-term differentiation to find the MacLaurin series of the function

$$f(x) = \frac{x^3}{(2-x)^2}.$$

Give the open interval of convergence. Express your final answer in sigma notation $\sum_n a_n x^n$.

**20 Marks**

$$\frac{1}{2-x} = \frac{1}{2(1-x/2)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n$$

for $\left|\frac{x}{2}\right| < 1 \iff |x| < 2$.

$$\frac{1}{(2-x)^2} = \frac{d}{dx}\left(\frac{1}{2-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n\right)$$

$$= \sum_{n=1}^{\infty}\frac{n}{2^{n+1}}x^{n-1} = \sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^n. \quad \text{So}$$

$$f(x) = \frac{x^3}{(2-x)^2} = x^3\sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^n = \sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^{n+3}$$

or $\displaystyle\sum_{n=3}^{\infty}\frac{n-2}{2^{n-1}}x^n$ for $|x|<2$.

**6.**

Use series to evaluate the limit

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}.$$

5 Marks

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - x + \frac{1}{6}x}{x^5}$$

$$= \lim_{x\to 0} \frac{\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^5}$$

$$= \lim_{x\to 0} \left(\frac{1}{5!} - \frac{1}{7!}x^2 + \cdots\right)$$

$$= \frac{1}{5!} = \frac{1}{120}$$