

1. Find an equation for the plane which contains the following two lines given by parametric equations

$$x = 1 + t, y = -1 - t, z = -1 + 2t \quad \text{and} \quad x = -1 + 2t, y = -4 + 3t, z = -3 + 2t.$$

Solution: The normal vector for the plane is given by $\vec{d}_1 \times \vec{d}_2$, where $\vec{d}_1 = (1, -1, 2)$ and $\vec{d}_2 = (2, 3, 2)$ are direction vectors of the lines above. So we get

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & 3 & 2 \end{vmatrix} = (-8, 2, 5)$$

Since the plane contains the point $(1, -1, -1)$, we get $\vec{n} \cdot (1, -1, -1) = -15$.

Hence, the desired equation is $-8x + 2y + 5z = -15$.

cross (X) the correct answer:

A $2x + 2y + 2z = 0$

B $-8x - y - z = -15$

C $-8x + 2y + 5z = -15$ - **Correct**

D $x - y + 5z = 0$

E $2x + 3y + 2z = -15$

F $-8x + 2y + 5z = 0$

2. Find an equation of the plane passing through the points $P = (-3, 1, 2)$ and $Q = (1, 2, -2)$, and which is parallel to the z -axis.

Solution. Such a plane is parallel to the vectors $(1, 2, -2) - (-3, 1, 2) = (4, 1, -4)$ and $(0, 0, 1)$ (as it is parallel to the z -axis). Therefore, a normal vector is

$$\begin{vmatrix} i & j & k \\ 4 & 1 & -4 \\ 0 & 0 & 1 \end{vmatrix} = (1, -4, 0).$$

Since this plane passes through P we have $(-3, 1, 2) \cdot (1, -4, 0) = -7$. Hence, the equation is $(1)x + (-4)y = -7$.

cross (X) the correct answer:

A $x + y + z = 0$

B $-x + y + 2z = 1$

C $x - 4y = 3$

D $x + 2y - 2z = 0$

E $x - 4y = -7$ - **Correct**

F $-x + y + 2z = 0$

3. Find an equation of the plane which passes through the point $(-5, 2, -1)$ and which is perpendicular to the line whose parametric equation is:

$$x = 7 - 4t, \quad y = 2 + 2t, \quad z = -3 + t.$$

Solution: A normal vector \vec{n} for the plane will be the direction vector for this line which is $(-4, 2, 1)$. Since the plane contains the point $(-5, 2, -1)$, we get $\vec{n} \cdot (-5, 2, -1) = 23$. Hence, an equation for the plane is $-4x + 2y + z = 23$.

cross (X) the correct answer:

A $-4x + 2y + z = 23$ - **Correct**

B $-4x + 2y + z = -23$

C $2x - 4y + z = -10$

D $2x - 4y + z = 10$

E $7x + y - 3z = -71$

F $-4x + 2y + z = -28$

4. Parametric equation for the line containing the points $(-8, 6, 6)$ and $(-2, 2, 3)$ is:

Solution: A direction vector for such a line is the difference $(-8, 6, 6) - (-2, 2, 3) = (-6, 4, 3)$. There are two lines above with this direction vector, namely A and E. But only E passes through both given points, so the correct answer is E.

cross (X) the correct answer:

A $x = 4 - 6t, y = -2 + 4t, z = 1 + 3t$

B $x = -1 - 6t, y = 3 + 4t, z = 6 + t$

C $x = -2 + 4t, y = 2 - 2t, z = 3$

D $x = -1 - 6t, y = 1 - t, z = 4 + 3t$

E $x = -2 - 6t, y = 2 + 4t, z = 3 + 3t$ - **Correct**

F Such a line does not exist

5. Which two of the following are vector parametric descriptions for the plane with equation $x - 2y + z = 4$?

I. $v = (0, 0, 0) + s(2, 1, 0) + t(0, 1, 2)$

II. $v = (0, 0, 4) + s(-1, 0, 1) + t(2, 1, 0)$

III. $v = (0, -2, 0) + s(-1, 0, 1) + t(0, 1, 2)$

IV. $v = (0, 0, 4) + s(1, 1, 1) + t(1, 0, 1)$

Solution: Since the plane does not contain $(0, 0, 0)$, we can exclude I.

When $v = P + sv_1 + tv_2$ is the vector parametric equation of a plane, the vectors v_1 and v_2 must be perpendicular to any normal vector; in this case it is $(1, -2, 1)$.

Simple checks show that $(-1, 0, 1)$, $(2, 1, 0)$, $(1, 1, 1)$, $(0, 1, 2)$ are perpendicular to $(1, -2, 1)$, but $(1, 0, 1)$ is not. So IV. can not be correct.

cross (X) the correct answer:

A I. and II.

B I. and III.

C I. and IV.

D II. and III. - **Correct**

E II. and IV.

F III. and IV.

6. Which of the vectors below is perpendicular (orthogonal) to both $(1, 2, -1)$ and $(1, 3, 5)$?

Solution: Such a vector will be parallel to

$$(1, 2, -1) \times (1, 3, 5) = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 1 & 3 & 5 \end{vmatrix} = (13, -6, 1).$$

cross (X) the correct answer:

A $(4, 9, 3)$

B $(-13, -6, 1)$

C $(1, -6, 13)$

D $(-6, 13, 1)$

E $(-13, 6, -1)$ - **Correct**

F None of the above

7. If $A = (2, 4, 1)$, $B = (3, 0, 9)$ and $C = (1, 4, 0)$, find the angle $\angle BAC$.

Solution: We find $\vec{u} = C - A = (1, 4, 0) - (2, 4, 1) = (-1, 0, -1)$ and $\vec{v} = B - A = (1, -4, 8)$. Let $\theta = \angle BAC$. Then

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} = \frac{(-1, 0, -1) \cdot (1, -4, 8)}{\sqrt{2}\sqrt{81}} = \frac{-9}{9\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Hence, $\theta = 3\pi/4$.

cross (X) the correct answer:

A $\pi/6$

B $\pi/4$

C $\pi/3$

D $\pi/2$

E $4\pi/3$

F $3\pi/4$ - Correct

8. Let L be the line passing through the points $P = (-1, 1, 0)$ and $Q = (-3, 4, 2)$. The point of intersection of L with the plane given by the equation $x + y - z = 1$ is:

Solution: The line L has the direction vector $(-3, 4, 2) - (-1, 1, 0) = (-2, 3, 2)$. So it is given by the following parametric equation

$$x = (-1) + (-2)t, \quad y = 1 + 3t, \quad z = 0 + 2t. \quad (*)$$

This line intersects $x + y - z = 1$ when

$$((-1) + (-2)t) + (1 + 3t) - (0 + 2t) = 1$$

which implies $(-1) + (-1)t = 0$, hence, $t = -1$.

Substituting into (*) we obtain the point of intersection $(1, -2, -2)$.

cross (X) the correct answer:

A $(1, 0, 0)$

B $(0, -1, -1)$

C $(1, -2, -2)$ - **Correct**

D $(-1, 0, -1)$

E $(\frac{1}{2}, \frac{1}{2}, 0)$

F $(0, \frac{1}{2}, -\frac{1}{2})$

9. If $\vec{u} = (1, 3, -1)$ and $\vec{v} = (1, 1, 1)$, then the orthogonal projection of \vec{u} on \vec{v} is:

Solution: We obtain

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \frac{(1, 3, -1) \cdot (1, 1, 1)}{(1)^2 + (1)^2 + (1)^2} (1, 1, 1) = \frac{3}{3} (1, 1, 1).$$

cross (X) the correct answer:

A $(1, 3, -1)$

B $4(1, 1, 1)$

C $4(1, 3, -1)$

D $(1, 1, 1)$ - **Correct**

E $-(1, 1, 1)$

F $-(1, 3, -1)$

10. Find the area of the triangle whose vertices are the points $A = (2, -1, 0)$, $B = (0, -1, 1)$ and $C = (1, 0, -2)$.

Solution: We find $\vec{AB} = (0, -1, 1) - (2, -1, 0) = (-2, 0, 1)$ and $\vec{AC} = (1, 0, -2) - (2, -1, 0) = (-1, 1, -2)$.

Then

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -2 & 0 & 1 \\ -1 & 1 & -2 \end{vmatrix} = (-1, -5, -2).$$

The area is

$$\frac{1}{2} \|\vec{u} \times \vec{v}\| = \frac{1}{2} \sqrt{(-1)^2 + (-5)^2 + (-2)^2} = \frac{1}{2} \sqrt{30}.$$

cross (X) the correct answer:

A 15

B 30

C $\sqrt{30}$

D $2\sqrt{30}$

E $\frac{1}{2}\sqrt{30}$ - Correct

F $\sqrt{15}$

The last page (use it for computations)

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, & \sin\left(\frac{\pi}{3}\right) &= \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\ \sin\left(\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, & \sin(0) &= \cos\left(\frac{\pi}{2}\right) = 0, & \sin\left(\frac{\pi}{2}\right) &= \cos(0) = 1.\end{aligned}$$