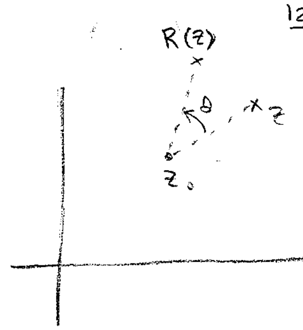
* Rotation with Centre z_0 and angle $0 \leqslant \theta \leqslant 2\pi$:

$$(R(z) - z_0) = e^{i\theta}(z - z_0) \Rightarrow \underbrace{R(z) = e^{i\theta} z + z_0 - e^{i\theta} z_0}$$

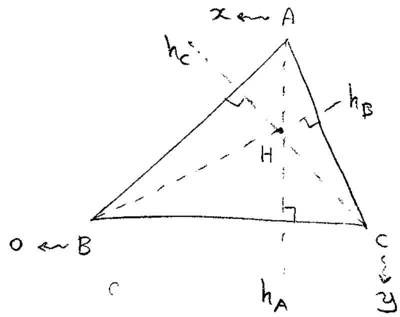


* Conclusion Every reflection, translation, and rotation of $\mathbb{E}^2$

Can be written as $f(z) = \alpha z + \beta$ or $f(z) = \alpha\bar{z} + \beta$ where

$\alpha, \beta \in \mathbb{C}$ and $|\alpha| = 1$.

Applications of Complex numbers in Euclidean geometry



* Prove that the altitudes of a triangle are concurrent.

Solution Vertices : $0, x, y \in \mathbb{C}$

Equations of altitudes: h_A : * $z = x + e^{\pi i/2}(y - 0)t$

$= x + iyt \quad (t \in \mathbb{R})$

h_C : * $z = y + ixt' \quad (t' \in \mathbb{R})$

h_B : * $z = i(x-y)t'' \quad (t'' \in \mathbb{R})$

$$\underbrace{x + iyt = y + ixt'}_{\text{intersection of } h_A \& h_C} \Rightarrow iyt - ixt' = y - x \Rightarrow \begin{cases} iyt - ixt' = y - x \\ -i\bar{y}t + i\bar{x}t' = \bar{y} - \bar{x} \end{cases}$$

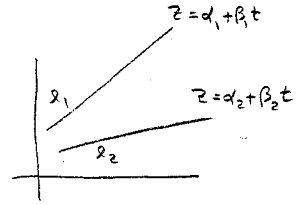
$$\underset{\text{Cramer's rule}}{\Rightarrow} t = \frac{\begin{vmatrix} y-x & -ix \\ \bar{y}-\bar{x} & i\bar{x} \end{vmatrix}}{\begin{vmatrix} iy & -ix \\ -i\bar{y} & i\bar{x} \end{vmatrix}} = \frac{(y-x)(i\bar{x}) + ix(\bar{y}-\bar{x})}{-\bar{x}y + x\bar{y}}$$

$$\Rightarrow z = x + iyt = \frac{1}{-\bar{x}y + x\bar{y}}\left(-x\bar{x}y + x^2\bar{y} + (x-y)\bar{x}y - xy\bar{y} + x\bar{x}y\right)$$

$$= \frac{1}{-\bar{x}y + x\bar{y}}(x-y)(\bar{x}y + x\bar{y}) = (x-y) \cdot \frac{\bar{x}y + x\bar{y}}{-\bar{x}y + x\bar{y}}$$

$x\bar{y} + \bar{x}y \in \mathbb{R}$ and $-\bar{x}y + x\bar{y} \in \mathbb{R}i \Rightarrow z = i(x-y)t''$ for some $t'' \in \mathbb{R}$. $\blacksquare$

Example Using complex numbers show that the composition of two reflections in parallel lines ℓ_1 and ℓ_2 is a translation.



$$\left.\begin{aligned} R_1(z) &= \beta_1^2 \bar{z} + \alpha_1 - \bar{\alpha}_1 \beta_1^2 \\ R_2(z) &= \beta_2^2 \bar{z} + \alpha_2 - \bar{\alpha}_2 \beta_2^2 \end{aligned}\right\} \Rightarrow R_1 \circ R_2(z) = R_1(\beta_2^2 \bar{z} + \alpha_2 - \bar{\alpha}_2 \beta_2^2)$$

$$= \beta_1^2 \overline{(\beta_2^2 \bar{z} + \alpha_2 - \bar{\alpha}_2 \beta_2^2)} + \alpha_1 - \bar{\alpha}_1 \beta_1^2$$

$$= \beta_1^2 \bar{\beta}_2^2 z + \beta_1^2 \bar{\alpha}_2 - \beta_1^2 \alpha_2 \bar{\beta}_2^2 + \alpha_1 - \bar{\alpha}_1 \beta_1^2 \qquad (|\beta_1| = |\beta_2| = 1)$$

$\ell_1 \parallel \ell_2 \Rightarrow$ We can assume $\beta_1 = \beta_2 \Rightarrow \beta_1 \bar{\beta}_2 = 1 \Rightarrow R_1 \circ R_2(z) = z + (\beta_1^2 \bar{\alpha}_2 - \alpha_2 + \alpha_1 - \bar{\alpha}_1 \beta_1^2) = z + \underset{\text{Constant}}{C}$ (translation)

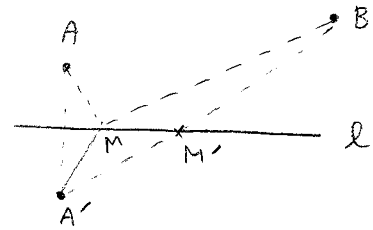
* Bonus Question Let $f : \mathbb{E}^2 \to \mathbb{E}^2$ be a function satisfying the following statement:

$$\forall_{P,Q \in \mathbb{E}^2} : \; d(P,Q) = 1 \Rightarrow d(f(P), f(Q)) = 1.$$

Prove that f is an isometry of $\mathbb{E}^2$.

Applications of isometries

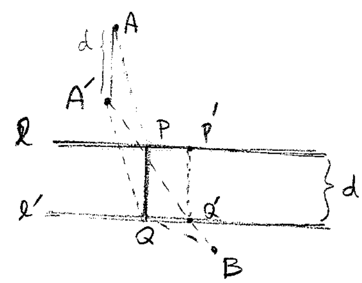
Example You are at a point A. You want to walk to the river (line ℓ), then to point B. What is the shortest possible path?



Solution Reflect A. Then $AM + MB = A'M + MB \geq A'B = A'M' + M'B$

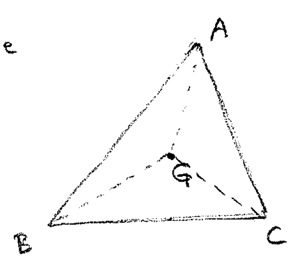
Therefore the shortest path is $A \to M' \to B$ where M' is the intersection of the line $A'B$ with the line ℓ.

Example We want to build a bridge on a river of width d to connect the towns A and B. Where should we build it to minimize the length of the roads?

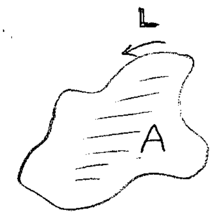


Solution Translate A. Then: $AP + QB = A'Q + QB \geq A'Q' + Q'B$ and therefore the optimal bridge is $P'Q'$ where Q' is the intersection of $A'B$ with ℓ'.

Bonus Question Three towns A, B, C need electricity. Determine the position of the generator G such that the total length of power lines is minimized.

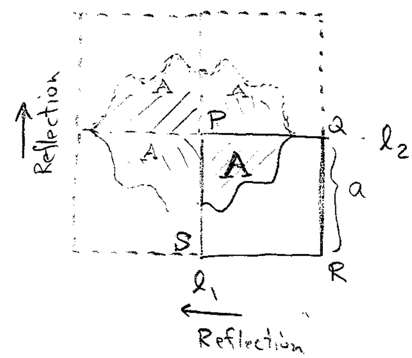


Theorem If a non-intersecting closed curve of length L encloses a domain of area A, then $4\pi A \leqslant L^2$. Equality holds if and only if the curve is a circle.



Example Find the shortest wall between the top and the left borders of a square-shaped piece of land which divides it into two parts of equal area.

Solution We claim that the wall should be along a quarter of a circle with radius $\sqrt{\frac{2}{\pi}}\,a$ and centre P.



1* Let L be the length of the wall

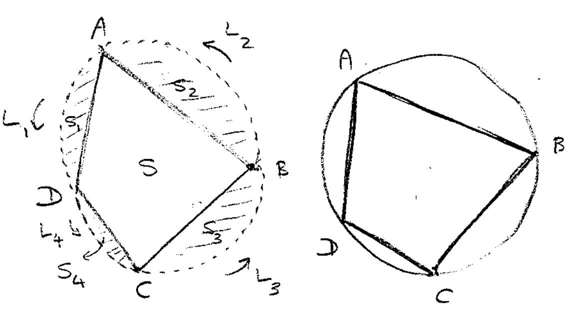
* Reflect the picture along l_1 and then l_2.

* We obtain a closed non-intersecting curve of length $\underline{\underline{4L}}$ enclosing area $\underline{\underline{4A}} = 2a^2$.

* (isoperimetric inequality): $4\pi(4A) \leqslant (4L)^2 \Rightarrow 16\pi\left(\frac{a^2}{2}\right) \leqslant 16L^2 \Longrightarrow L \geqslant \sqrt{\frac{\pi}{2}}\,a$

* equality takes place for the quater of the circle with radius $\sqrt{\frac{2}{\pi}}\,a$ with centre $A = (0,0)$.

Example Which quadrilateral with side lengths a, b, c, d (in the same order) encloses the largest area?



Solution There is a unique quadrilateral with side lengths a, b, c, d whose vertices lie on a circle. Fixing the arcs on the sides, we observe that

$$S + S_1 + S_2 + S_3 + S_4 \leqslant (L_1 + L_2 + L_3 + L_4)^2 / 4\pi \quad \text{(isoperimetric inequality)}$$