Introduction to Complex numbers

$\mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$

$i = "\sqrt{-1}" \Rightarrow i^2 = -1$

$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$

<u>Addition</u> : $(a+bi) + (a'+b'i) = (a+a') + (b+b')i$

<u>Example</u> $(2-3i) + (-1+i) = (2+(-1)) + ((-3)+1)i = 1-2i$.

<u>multiplication</u> : $(a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac-bd) + (ad+bc)i$

<u>Example</u> $(2-3i)(1+i) = (2-(-3)) + (2-3)i = 5-i$

<u>Complex Conjugate</u> : $z = a+bi \Rightarrow \bar{z} = a-bi$

<u>Example</u> $\overline{2-3i} = 2+3i$, $\bar{i} = -i$.

<u>Norm</u> $z = a+bi \Rightarrow |z| = (z \cdot \bar{z})^{\frac{1}{2}} = ((a+bi)(a-bi))^{\frac{1}{2}} = (a^2+b^2)^{\frac{1}{2}}$

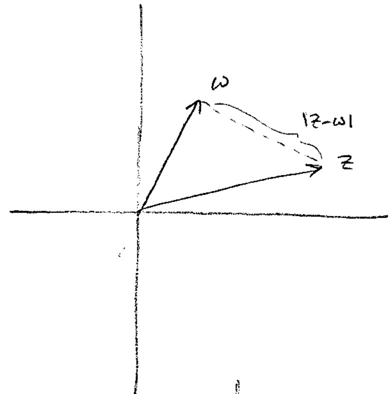
<u>Example</u> $|1-2i| = \sqrt{1+(-2)^2} = \sqrt{5}$.

<u>Inversion</u> $\frac{1}{1-2i} = \frac{1+2i}{(1-2i)(1+2i)} = \frac{1+2i}{1+4} = \frac{1}{5}(1+2i) = \frac{1}{5} + \frac{2}{5}i$.

Complex numbers and isometries of $\mathbb{E}^2$

- $\mathbb{E}^2 \simeq \mathbb{C} \qquad (a,b) \longleftrightarrow a+bi$

$*$ $z, w \in \mathbb{C} \Rightarrow d(z,w) = |z-w|$

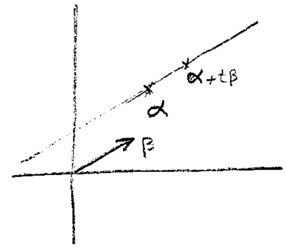


$*$ Dot product : $\vec{z} \bullet \vec{w} = \frac{1}{2}(|z+w|^2 - |z|^2 - |w|^2)$

$$= \frac{1}{2}\left((z+w)(\bar{z}+\bar{w}) - z\bar{z} - w\bar{w}\right)$$

$$= \frac{1}{2}(z\bar{w} + \bar{z}w)$$

Example $\left.\begin{array}{l}\vec{z} = 1+i \\ \vec{w} = 1-i\end{array}\right\} \Rightarrow \vec{z}\cdot\vec{w} = \frac{1}{2}\left((1+i)(1+i) + (1-i)(1-i)\right) = 0.$

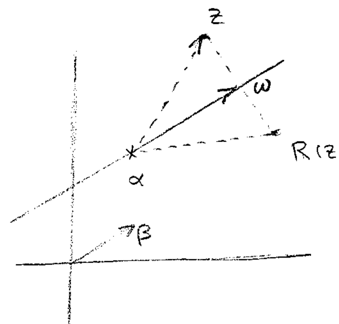


$*$ Lines : $z = \alpha + \beta t \qquad$ $\alpha \in \mathbb{C}$ fixed

$\beta \in \mathbb{C}$, $|\beta| = 1$, fixed

$t \in \mathbb{R}$ parameter

I. Reflection in the line $z = \alpha + \beta t$, $|\beta| = 1$



$*$ $(w - \alpha) = ((z-\alpha)\bullet\beta)\,\beta = \frac{1}{2}\left((z-\alpha)\bar{\beta} + \overline{(z-\alpha)}\beta\right)\beta$

$$= \frac{1}{2}(z-\alpha) + \frac{1}{2}\overline{(z-\alpha)}\,\beta^2$$

$*$ $(z-\alpha) + (R(z) - \alpha) = 2(w-\alpha)$

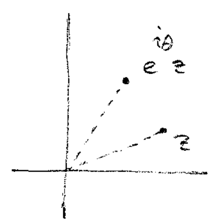
$\Rightarrow R(z) = 2(w-\alpha) - z + 2\alpha = (z-\alpha) + \overline{(z-\alpha)}\beta^2 - z + 2\alpha$

$\Rightarrow \underline{R(z) = \beta^2 \bar{z} + \alpha - \bar{\alpha}\beta^2}$ $\qquad \Rightarrow$ Every reflection is of the form $z \mapsto \gamma_1 \bar{z} + \gamma_2$; $\gamma_1, \gamma_2 \in \mathbb{C}$, $|\gamma_1| = 1$.

II. Translation by $\alpha \in \mathbb{C}$

$$z \longmapsto z + \alpha \qquad \alpha \in \mathbb{C}$$

III Rotations



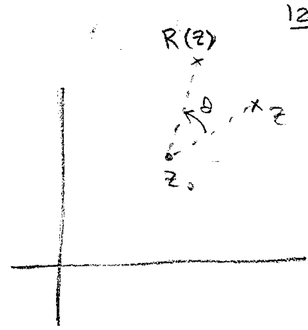
$*$ Rotation with Centre $z_0 = 0$ and angle $0 \le \theta \le 2\pi$:

$$z \longmapsto e^{i\theta} z$$

$z = x + iy \Rightarrow e^{i\theta} z = (\cos\theta + i\sin\theta)(x+iy) = (x\cos\theta - y\sin\theta) + (x\sin\theta + y\cos\theta)\,i$

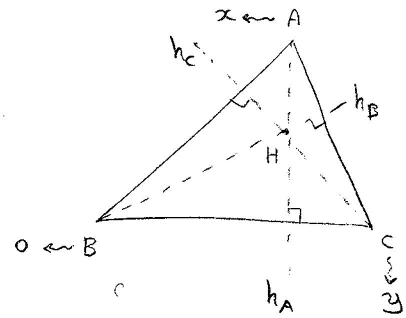
* Rotation with Centre z_0 and angle $0 \leq \theta \leq 2\pi$:

$$R(z) - z_0 = e^{i\theta}(z - z_0) \Rightarrow R(z) = e^{i\theta} z + z_0 - e^{i\theta} z_0$$



* **Conclusion** Every reflection, translation, and rotation of $\mathbb{E}^2$ Can be written as $f(z) = \alpha z + \beta$ or $f(z) = \alpha \bar{z} + \beta$ where $\alpha, \beta \in \mathbb{C}$ and $|\alpha| = 1$.

Applications of Complex numbers in Euclidean geometry



* Prove that the altitudes of a triangle are concurrent.

Solution Vertices : $0, x, y \in \mathbb{C}$

Equations of altitudes:

h_A : * $z = x + e^{\pi i/2}(y - 0)t$
$= x + iyt \quad (t \in \mathbb{R})$

h_C : * $z = y + ixt' \quad (t' \in \mathbb{R})$

h_B : * $z = i(x-y)t'' \quad (t'' \in \mathbb{R})$

$\underbrace{x + iyt = y + ixt'}_{\text{intersection of } h_A \,\&\, h_C} \Rightarrow iyt - ixt' = y - x \Rightarrow \begin{cases} iyt - ixt' = y - x \\ -i\bar{y}t + i\bar{x}t' = \bar{y} - \bar{x} \end{cases}$

$$\underset{\text{Cramer's rule}}{\Rightarrow} t = \frac{\begin{vmatrix} y-x & -ix \\ \bar{y}-\bar{x} & i\bar{x} \end{vmatrix}}{\begin{vmatrix} iy & -ix \\ -i\bar{y} & i\bar{x} \end{vmatrix}} = \frac{(y-x)(i\bar{x}) + ix(\bar{y}-\bar{x})}{-\bar{x}y + x\bar{y}}$$

$$\Rightarrow z = x + iyt = \frac{1}{-\bar{x}y + x\bar{y}}\left(-x\bar{x}y + x^2\bar{y} + (x-y)\bar{x}y - xy\bar{y} + x\bar{x}y\right)$$

$$= \frac{1}{-\bar{x}y + x\bar{y}}(x-y)(\bar{x}y + x\bar{y}) = (x-y) \cdot \frac{\bar{x}y + x\bar{y}}{-\bar{x}y + x\bar{y}}$$

$x\bar{y} + \bar{x}y \in \mathbb{R}$ and $-\bar{x}y + x\bar{y} \in \mathbb{R}i \Rightarrow z = i(x-y)t''$ for some $t'' \in \mathbb{R}$. ▨