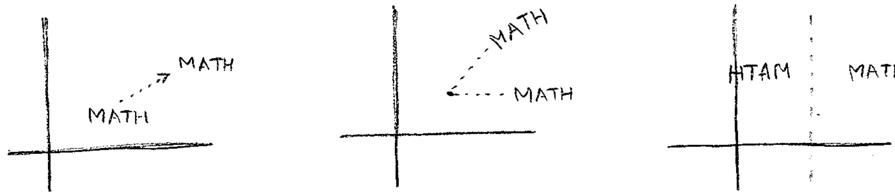
Observations

* Translations and rotations are orientation-preserving; reflections are <u>not</u>.

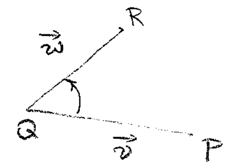


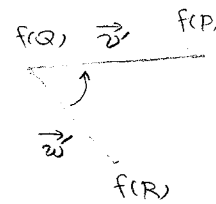
* <u>Composition of isometries</u> If $f: \mathbb{E}^n \longrightarrow \mathbb{E}^n$ and $g: \mathbb{E}^n \longrightarrow \mathbb{E}^n$ are isometries then $g \circ f : \mathbb{E}^n \longrightarrow \mathbb{E}^n$ is also an isometry.

<u>Proof</u> $P, Q \in \mathbb{E}^n \Longrightarrow d(g(f(P)), g(f(Q))) = d(f(P), f(Q)) = d(P,Q)$. $\blacksquare$

* <u>Angle-preservation</u> If $f: \mathbb{E}^n \longrightarrow \mathbb{E}^n$ is an isometry then for every three points $P, Q, R \in \mathbb{E}^n$ we have $\angle PQR = \angle f(P) f(Q) f(R)$.

<u>Proof</u> * Let $\vec{v} = \overrightarrow{QP}$, $\vec{w} = \overrightarrow{QR}$, $\vec{v}' = \overrightarrow{f(Q)f(P)}$
$\vec{w}' = \overrightarrow{f(Q)f(R)}$





* $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\angle PQR)$

* $\vec{v}' \cdot \vec{w}' = \|\vec{v}'\| \cdot \|\vec{w}'\| \cdot \cos(\angle f(P) f(Q) f(R))$

* $\angle PQR = \angle f(P) f(Q) f(R) \iff \cos(\angle PQR) = \cos(\angle f(P) f(Q) f(R))$

* To complete the proof, we show:
  (a) $\|\vec{v}\| = \|\vec{v}'\|$
  (b) $\|\vec{w}\| = \|\vec{w}'\|$
  (c) $\vec{v} \cdot \vec{w} = \vec{v}' \cdot \vec{w}'$

(a) $\|\vec{v}\| = d(P,Q) = d(f(P), f(Q)) = \|\vec{v}'\|$.

(b) $\|w\| = d(R,Q) = d(f(R), f(Q)) = \|\vec{w}'\|$.

(c) $\|\vec{v} - \vec{w}\|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\vec{v} \cdot \vec{w} \Rightarrow \vec{v} \cdot \vec{w} = -\frac{1}{2}(\|\vec{v} - \vec{w}\|^2 - \|\vec{v}\|^2 - \|\vec{w}\|^2)$

Similarly, $\vec{v}' \cdot \vec{w}' = -\frac{1}{2}(\|\vec{v}' - \vec{w}'\|^2 - \|v'\|^2 - \|w'\|^2)$

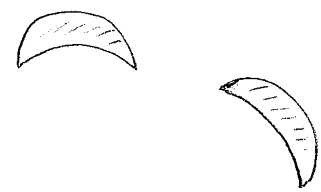
It follows that $\vec{v}\cdot\vec{w} = -\frac{1}{2}\left(\|\vec{v}-\vec{w}\|^2 - \|v\|^2 - \|w\|^2\right) = -\frac{1}{2}\left(d(P,R)^2 - d(P,Q)^2 - d(R,Q)^2\right)$

$$= -\frac{1}{2}\left(d(f(P),f(R))^2 - d(f(P),f(Q))^2 - d(f(R),f(Q))^2\right)$$

$$= -\frac{1}{2}\left(\|\vec{v}'-\vec{w}'\|^2 - \|\vec{v}'\|^2 - \|w'\|^2\right) = \vec{v}'\cdot\vec{w}' .$$

∎

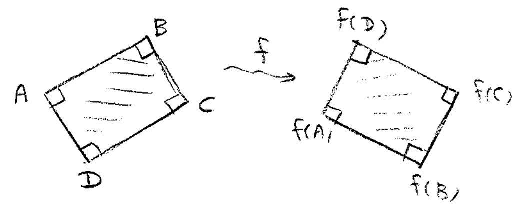
* Area - preservation If $f : \mathbb{E}^2 \to \mathbb{E}^2$ is an isometry and $R \subseteq \mathbb{E}^2$ is a region then $\text{Area}(f(R)) = \text{Area}(R)$.



Proof We prove the statement when R is a rectangle.
The general case follows by "appoximating the area of R" by tiling it with small rectangles.

* If R is a rectangle then $f(R)$ is a rectangle.

* $\text{Area}(R) = \overline{AB}\cdot\overline{AD} = (\overline{f(A)f(B)})\cdot(\overline{f(A)f(D)})$
$= \text{Area}(f(R))$.



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* Composition of two translations is a translation

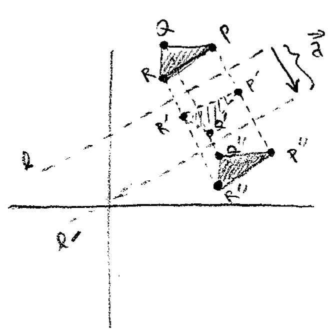
$$\vec{v} \longmapsto \vec{v}+\vec{a} \longmapsto \vec{v}+\vec{a}+\vec{b} .$$

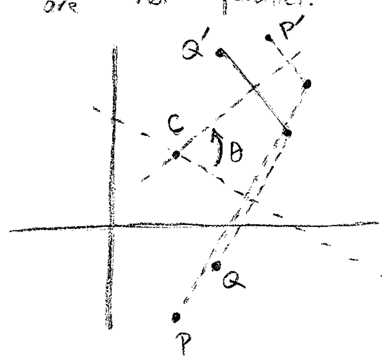
* Composition of two reflections in lines ℓ, ℓ' is ...

  * a translation if ℓ and ℓ' are parallel.

  * a rotation if ℓ and ℓ' are not parallel.

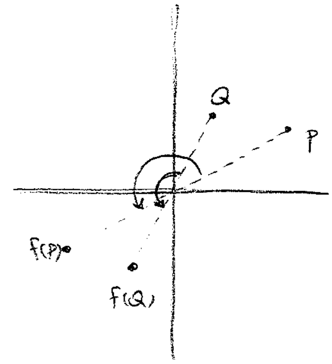
$\vec{v} \longmapsto \vec{v}+2\vec{a}$





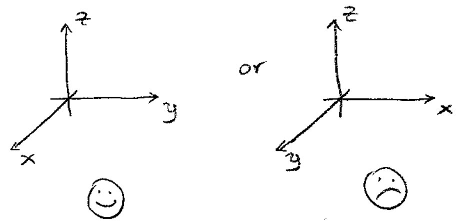
Rotation with Centre C and angle 2θ.

* A central symmetry is the same as a rotation with angle π.



* **Conclusion** We can obtain translations and rotations by composing reflections.

Orientation in $\mathbb{R}^3$



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Right-hand rule: Thumb $\rightsquigarrow$ x-axis
index $\rightsquigarrow$ y-axis
middle $\rightsquigarrow$ z-axis

Definition A triple $(\vec{u},\vec{v},\vec{w})$ of vectors in $\mathbb{R}^3$ is called positively oriented if $\det(\vec{u}\,|\,\vec{v}\,|\,\vec{w}) > 0$. (Otherwise $(\vec{u},\vec{v},\vec{w})$ is called negatively oriented.)

Example $\left(\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$ is positively oriented because $\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix} = 1 > 0.$

$\left(\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$ is negatively oriented because $\begin{vmatrix}0&1&0\\1&0&0\\0&0&1\end{vmatrix} = -1 < 0.$

* More generally, an n-tuple $(\vec{v}_1,\dots,\vec{v}_n)$ of linearly independent vectors in $\mathbb{R}^n$ is called positively oriented if $\det(\vec{v}_1\,|\cdots|\,\vec{v}_n) > 0$ (and negatively oriented if $\det(\vec{v}_1|\cdots|\vec{v}_n)<0$)

* A linear map $T:\mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$ (where A is an $n\times n$ matrix) is called orientation preserving (orientation reversing) if $\det(A) > 0$ ($\det(A) < 0$).

Bonus Question

Compute the angle x in the picture. Justify your answer (a measurement by a proctractor is not accepted)