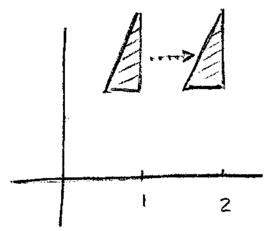
§1.2.1 <u>Definition</u> Let $n \geq 1$ be an integer. A function $f : \mathbb{E}^n \longrightarrow \mathbb{E}^n$ is called an <u>isometry</u> if for every $P, Q \in \mathbb{E}^n$, $d(f(P), f(Q)) = d(P, Q)$.

<u>Example</u> Let $f : \mathbb{E}^2 \longrightarrow \mathbb{E}^2$, $f(x,y) = (x+1, y)$. Then f is an isometry

$$\left.\begin{array}{l} P=(x,y) \\ Q(x',y') \end{array}\right\} \Rightarrow d(f(P), f(Q)) = d((x+1,y),(x'+1,y')) = \left(((x+1)-(x'+1))^2 + (y-y')^2\right)^{\frac{1}{2}}$$

$$= \left((x-x')^2 + (y-y')^2\right)^{\frac{1}{2}} = d(P,Q).$$

The above map is a <u>translation</u> parallel to the x-axis

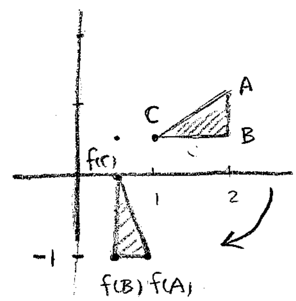


<u>Example</u> Let $f : \mathbb{E}^2 \longrightarrow \mathbb{E}^2$, $f(x,y) = (y, -x+1)$ is an isometry

$$\left.\begin{array}{l} P=(x_P,y_P) \\ Q=(x_Q,y_Q) \end{array}\right\} \Rightarrow d(f(P), f(Q)) = d((y_P, -x_P+1), (y_Q, -x_Q+1))$$

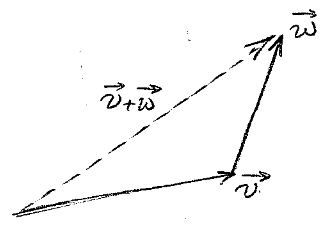
$$= \left((y_P - y_Q)^2 + ((-x_P+1) - (-x_Q+1))^2\right)^{\frac{1}{2}}$$

$$= \left((y_P - y_Q)^2 + (x_P - x_Q)^2\right)^{\frac{1}{2}} = d(P,Q)$$



The above map is a clockwise rotation of 90 degrees with centre $(\frac{1}{2}, \frac{1}{2})$.

<u>The Triangle Inequality</u>

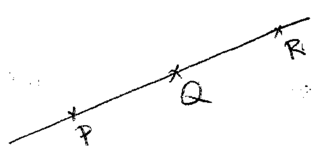


* For every two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ we have:

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|.$$

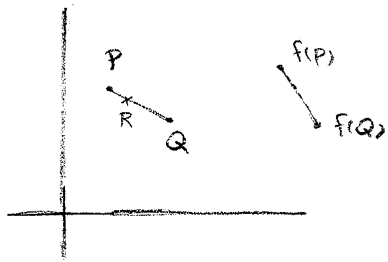
* Consequently, if P, Q, R are three points in $\mathbb{E}^n$ then

$$d(P,R) \leq d(P,Q) + d(Q,R)$$



Equality holds if and only if P, Q, R are collinear and Q is between P and R.

<u>Proposition</u> Let $f : \mathbb{E}^n \longrightarrow \mathbb{E}^n$ be an isometry, and $P, Q \in \mathbb{E}^n$ two points. Then f maps the line segment PQ to the line segment $f(P)f(Q)$.

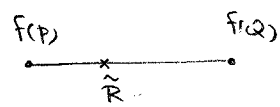


<u>Proof</u> <u>Step 1</u> Let R be on the line segment PQ.

$$d(P,R) + d(R,Q) = d(P,Q) \Rightarrow d(f(P), f(R)) + d(f(R), f(Q)) = d(f(P), f(Q))$$

$\Rightarrow f(P), f(Q), f(R)$ are collinear and $f(R)$ lies between $f(P), f(Q)$.

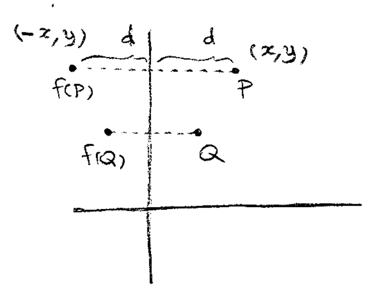
<u>Step 2</u> Next we show that if $\tilde{R}$ is on the line segment between $f(P)$ and $f(Q)$, then $\tilde{R} = f(R)$ for some R between P and Q.



* $0 \leq d(f(P), \tilde{R}) \leq d(f(P), f(Q)) = d(P,Q)$
* Therefore we can choose R on PQ such that $d(P,R) = d(f(P), \tilde{R})$.
* By Step 1, $f(R)$ is on the line segment $f(P)f(Q)$.
* $d(f(P), \tilde{R}) = d(P,R) = d(f(P), f(R)) \Rightarrow \tilde{R} = f(R)$.

§1.2.2 <u>Definition</u> Two sets $S, T \subseteq \mathbb{E}^n$ are called <u>Congruent</u> if there exists an isometry $f : \mathbb{E}^n \longrightarrow \mathbb{E}^n$ such that $f(S) = T$.

<u>Examples of isometries of $\mathbb{E}^n$</u>



I. <u>Reflections</u>

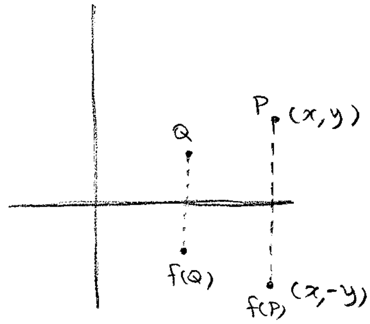
Reflection in the y-axis $\rightsquigarrow$ $f : \mathbb{E}^2 \longrightarrow \mathbb{E}^2$ $\quad f(x,y) = (-x, y)$

$$\left.\begin{array}{l} P = (x,y) \\ Q = (x',y') \end{array}\right\} \Rightarrow d(f(P), f(Q)) = d((-x,y), (-x',y'))$$

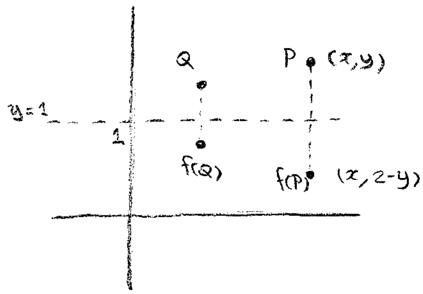
$$= \left((-x-(-x'))^2 + (y-y')^2\right)^{\frac{1}{2}}$$

$$= \left((x-x')^2 + (y-y')^2\right)^{\frac{1}{2}} = d(P,Q).$$

* reflection in the x-axis $\rightsquigarrow$ $f: \mathbb{E}^2 \longrightarrow \mathbb{E}^2$ $\quad f(x,y) = (x,-y)$

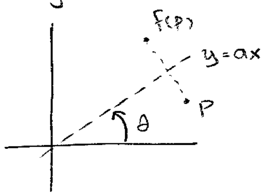


* reflection in the line $y=1$. $\rightsquigarrow$ $f: \mathbb{E}^2 \longrightarrow \mathbb{E}^2$ $\quad f(x,y) = (x, 2-y)$

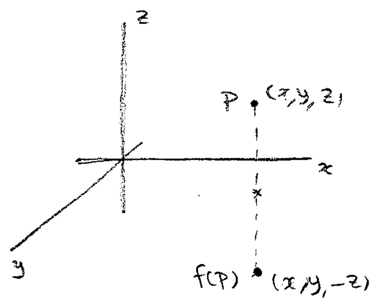


* reflection in the line $y=ax$ $\quad a=\tan(\theta)$ $\rightsquigarrow$

$$f(x,y) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



reflection in the xy-plane $\rightsquigarrow$ $f: \mathbb{E}^3 \longrightarrow \mathbb{E}^3$ $\quad f(x,y,z) = (x,y,-z)$



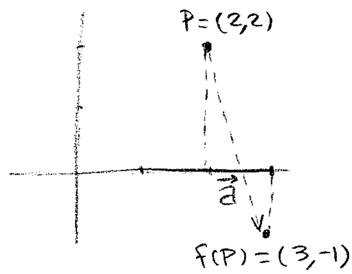
II. Translations

Definition If $\vec{a} \in \mathbb{E}^n$ is a fixed vector, the map

$$f: \mathbb{E}^n \longrightarrow \mathbb{E}^n \qquad f(\vec{v}) = \vec{v} + \vec{a}$$

is called a translation by the displacement vector $\vec{a}$.

Example $\quad f: \mathbb{E}^2 \longrightarrow \mathbb{E}^2 \quad f(x,y) = (x+1, y-3)$ is a translation by $\vec{a} = (1,-3)$.



III. Rotations (with center (0,0) and angle θ)

$$f: \mathbb{E}^2 \longrightarrow \mathbb{E}^2$$

$$f(x,y) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$