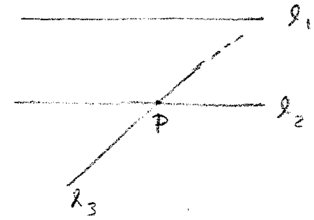
* Let ℓ_1, ℓ_2, ℓ_3 be distinct lines such that ℓ_1 and ℓ_2 are parallel and ℓ_2 intersects ℓ_3. Then ℓ_1 intersects ℓ_3 as well. (Follow from P5)



Development of geometry

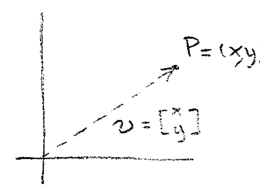
* Analytic geometry (Descortes ~ 1637) : using coordinate systems.

* Projective geometry (Desargues ~ 1648) : using ideas from perspective, avoiding the concept of length, using collinearity and transversality.

* Non-Euclidean geometry (Gauss, Lobatchevsky, Riemann, Bolyai, ... ~ 19^{th} century)

Basic ideas of analytic geometry

* Every point $P = (x_1, \dots, x_n) \in \mathbb{R}^n$ determines a vector $\vec{v} = (x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$



* $\mathbb{R}^n$ as a vector space :
  * Given two vectors $\vec{v} = (v_1, \dots, v_n)$ and $\vec{w} = (w_1, \dots, w_n)$, we define

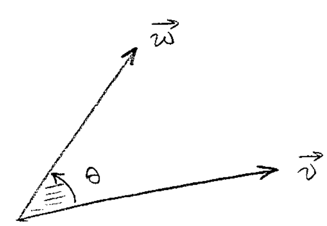
$$\vec{v} + \vec{w} = (v_1 + w_1, \dots, v_n + w_n)$$

$$c.\vec{v} = (cv_1, \dots, cv_n)$$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \dots + v_n w_n$$

* The <u>length</u> of $\vec{v} = (v_1, \dots, v_n)$ is defined by

$$\|\vec{v}\| = (v_1^2 + \dots + v_n^2)^{\frac{1}{2}} = (\vec{v} \cdot \vec{v})^{\frac{1}{2}}$$



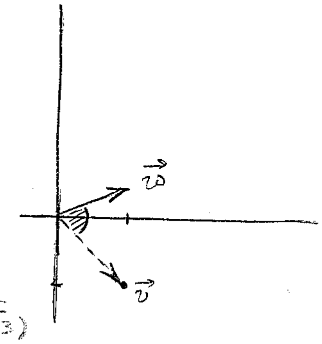
* For every two vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^n$ we have :

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$$

Example Compute the angle between $\vec{v} = (1,-1)$ and $\vec{w} = (1, 2-\sqrt{3})$

Solution $\vec{v}\cdot\vec{w} = (1)(1) + (-1)(2-\sqrt{3}) = \sqrt{3}-1$

$\|\vec{v}\| = \sqrt{1+1} = \sqrt{2}$, $\|\vec{w}\| = \sqrt{1+(2-\sqrt{3})^2} = \sqrt{8-4\sqrt{3}}$



$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|} = \frac{\sqrt{3}-1}{\sqrt{2}\cdot\sqrt{8-4\sqrt{3}}} = \frac{\sqrt{3}-1}{2\sqrt{2}(\sqrt{2-\sqrt{3}})}$$

$$= \frac{(\sqrt{3}-1)(\sqrt{2+\sqrt{3}})}{2\sqrt{2}\,\sqrt{4-3}} = \frac{\sqrt{(3+1-2\sqrt{3})(2+\sqrt{3})}}{2\sqrt{2}}$$

$$= \frac{\sqrt{4-2\sqrt{3}}\,\sqrt{2+\sqrt{3}}}{2\sqrt{2}} = \frac{\sqrt{2}\,\sqrt{2-\sqrt{3}}\,\sqrt{2+\sqrt{3}}}{2\sqrt{2}} = \frac{\sqrt{2}\,\sqrt{4-3}}{2\sqrt{2}} = \frac{1}{2}$$

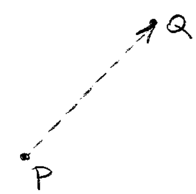
$$\Longrightarrow \cos(\theta) = \frac{1}{2} \Longrightarrow \theta = \frac{\pi}{3}.$$

Distance between two points in $\mathbb{R}^2$

$$P, Q \in \mathbb{R}^2 \Rightarrow d(P,Q) = \left((x_P-x_Q)^2 + (y_P-y_Q)^2\right)^{\frac{1}{2}} = \|\vec{PQ}\|$$

$P = (x_P, y_P)$

$Q = (x_Q, y_Q)$

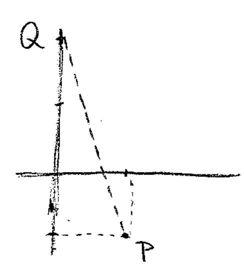


* Lines in $\mathbb{R}^2$: $\{(x,y)\in\mathbb{R}^2 : ax+by+c=0\}$

* Circles in $\mathbb{R}^2$: $\{(x,y)\in\mathbb{R}^2 : (x-a)^2+(y-b)^2 = r^2\}$

Example The circle with radius 2 and centre $(0,-1)$ has equation $x^2+(y+1)^2 = 4$.

Example If $P = (1,-1)$ and $Q = (0,2)$ then $d(P,Q) = \left((1-0)^2+(-1-2)^2\right)^{\frac{1}{2}} = \sqrt{10}$



Distance in $\mathbb{R}^n$:

$$\left.\begin{array}{l} P = (x_1,\dots,x_n) \\ Q = (y_1,\dots,y_n) \end{array}\right\} \Rightarrow d(P,Q) = \left((x_1-y_1)^2 + \dots + (x_n-y_n)^2\right)^{\frac{1}{2}}$$

$\rightsquigarrow$ The vector space $\mathbb{R}^n$ endowed with the above distance is usually denoted by $\mathbb{E}^n$ and is called the *n-dimensional Euclidean space*.