MAT1330 A – Instructor: Elizabeth Maltais
Wednesday, November 13, 2019 : Test #2

Duration: 75 minutes

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Please read the following instructions carefully.

- You have 75 minutes to complete this exam.
- This is a closed book exam. Except for Faculty-approved calculators (models: Texas Instruments TI-30* and TI-34*, Casio FX-260* and Casio FX-300*), no notes, cell phones, smartwatches or related devices of any kind are permitted. All such devices, including cell phones, **must be stored in your bag under your desk for the duration of the exam**.
- Read each question carefully — you will save yourself time and grief later on.
- Questions 1 through 6 are multiple choice, worth a total of 6 points. **Record your answers to the multiple choice questions in the boxes provided.**
- Questions 7 through 10 are long answer, with number of points as indicated. **You must show your work, your work must be legible and well-justified, and you must record your answers in the spaces provided.**
- Where it is possible to check your work, do so.
- Good luck!

SOLUTIONS

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Question	1–6	7	8	9	10	Total
Maximum	6	3	2	3	6	20
Grade						

Multiple-choice questions: Only your final answer will be considered.

1. (1 point) Suppose f and g are two differentiable functions. Some values of $f(x)$, $f'(x)$, $g(x)$ and $g'(x)$ (for $x = 2, 3, 5$) are given in the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	5	-3	3	5
3	1	7	5	-1
5	-1	3	2	5

If $h(x) = f(g(x))$, what is the value of $h'(5)$?

$h(x)=f(g(x))$
$\Rightarrow h'(x) = f'(g(x))\cdot g'(x)$
$\Rightarrow h'(5) = f'(g(5))\cdot g'(5)$
$= f'(2)\cdot g'(5)$
$=(-3)(5)$
$=-15$

A. 15
B. 35
C. -5
D. 21
E. -15
F. -30

Your answer: E

2. (1 point) If $g(x) = \sqrt{e^{-2x} + 2x^2}$, then $g'(x)$ is

A. $\dfrac{1}{2\sqrt{-2e^{2x} + 4x}}$

B. $(-2e^{-2x} + 4x)\sqrt{e^{-2x} + 2x^2}$

C. $\dfrac{e^{-2x} + 2x^2}{2\sqrt{e^{-2x} + 2x^2}}$

D. $\dfrac{-e^{-2x} + 2x}{\sqrt{e^{-2x} + 2x^2}}$

E. $-e^{-x} + \sqrt{2}$

F. $\dfrac{1}{2\sqrt{e^{-2x} + 2x^2}}$

Your answer: D

$g(x) = (e^{-2x}+2x^2)^{1/2}$

$g'(x) = \frac{1}{2}(e^{-2x}+2x^2)^{-\frac{1}{2}}(-2e^{-2x}+4x)$

$= \dfrac{-2e^{-2x}+4x}{2\sqrt{e^{-2x}+2x^2}} = \dfrac{\cancel{2}(-e^{-2x}+2x)}{\cancel{2}\sqrt{e^{-2x}+2x^2}} = \dfrac{-e^{-2x}+2x}{\sqrt{e^{-2x}+2x^2}}$

3. (1 point) Which of the following is the derivative of $g(x) = \ln(e^{-2x}x^6 \sin(x)6^x)$?

A. $-2 + \frac{6}{x} + \cot(x) + \ln(6)$ **B.** $\frac{-2e^{-2x} + 6x^5 + \cos(x) + 6^x \ln(6)}{e^{-2x}x^6 \sin(x)6^x}$ **C.** $\frac{-2e^{-2x} + \cos(x)}{e^{-2x}x^6 \sin(x)6^x}$

D. $-1 + \cot(x)$ **E.** $-2 + \frac{6}{x} + \tan(x)$ **F.** $-2 + \ln(6) + 6x^5 + \sec(x)$

Your answer: A

$$g(x) = \ln(e^{-2x}x^6 \sin(x) \cdot 6^x)$$
$$= \ln(e^{-2x}) + \ln(x^6) + \ln(\sin x) + \ln(6^x)$$
$$= -2x + 6\ln(x) + \ln(\sin(x)) + x\ln(6)$$
$$\Longrightarrow g'(x) = -2 + 6\left(\frac{1}{x}\right) + \frac{1}{\sin(x)}(\cos(x)) + \ln(6)$$
$$= -2 + \frac{6}{x} + \cot(x) + \ln(6)$$

4. (1 point) Consider the curve implicitly defined by the equation

$$y^3 + y^2 = x^2 + 3.$$

What is the slope of the tangent line to the curve at the point $(3, 2)$?

A. $\frac{1}{8}$ **B.** $\frac{2}{7}$ **C.** $\frac{3}{8}$ **D.** $\frac{1}{7}$ **E.** $\frac{1}{5}$ **F.** $\frac{1}{16}$

Your answer: C

$$y^3 + y^2 = x^2 + 3$$
$$\Longrightarrow 3y^2 \cdot y' + 2y \cdot y' = 2x + 0$$
$$\Longrightarrow 3y^2 \cdot y' + 2y \cdot y' = 2x$$
$$\Longrightarrow y'(3y^2 + 2y) = 2x$$
$$\Longrightarrow y' = \frac{2x}{3y^2 + 2y}$$

@ $(3,2)$ $y' = \frac{2(3)}{3(2^2) + 2(2)} = \frac{6}{16} = \frac{3}{8}$

5. (1 point) If $f(x) = x^{(3^x)}$ for $x > 0$, then $f'(x) =$

A. $x^{3^x - 1} 3^x \ln(3)$ **B.** $x^{3^x} 3^x \left(\ln(3)\ln(x) + \frac{1}{x}\right)$ **C.** $x^{3^x} \left(\ln(x) + \frac{1}{x}\right)$

D. $x^{3^x} 3^x \ln(3)\ln(x)$ **E.** $x^{3^x} 3^x \ln(3)$ **F.** $x^{3^x} 3^x (\ln(3) + \ln(x))$

Your answer: B

$$f(x) = x^{(3^x)} = e^{\ln(x^{(3^x)})} = e^{3^x \ln(x)}$$

$$f'(x) = e^{3^x \ln(x)} \cdot \frac{d}{dx}\left[3^x \ln(x)\right]$$

$$= e^{3^x \ln(x)} \cdot \left(\ln(3) \cdot 3^x \ln(x) + 3^x \cdot \left(\frac{1}{x}\right)\right)$$

$$= x^{3^x} \cdot (3^x) \cdot \left(\ln(3) \cdot \ln(x) + \frac{1}{x}\right)$$

6. (1 point) Given that the Taylor polynomial of order 4 of a function f centred at the base point $a = 1$ is

$$T_4(x) = 3 + (x-1)^2 + \frac{1}{12}(x-1)^3 + \frac{1}{12}(x-1)^4$$

which one of the following statements is correct?

A. $f(1) = 3$, $f'''(1) = \frac{1}{4}$ and $f^{(4)}(1) = \frac{1}{2}$ **B.** $f(1) = 3$, $f'''(1) = 2$ and $f^{(4)}(1) = 2$

C. $f(1) = \frac{1}{3}$, $f'''(1) = \frac{1}{2}$ and $f^{(4)}(1) = \frac{1}{2}$ **D.** $f(1) = 3$, $f'''(1) = \frac{1}{3}$ and $f^{(4)}(1) = 6$

E. $f(1) = \frac{1}{2}$, $f'''(1) = \frac{1}{6}$ and $f^{(4)}(1) = 3$ **F.** $f(1) = 3$, $f'''(1) = \frac{1}{2}$ and $f^{(4)}(1) = 2$

Your answer: F

$$T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 \quad (a=1)$$

$$T_4(x) = 3 + 0(x-1) + (x-1)^2 + \frac{1}{12}(x-1)^3 + \frac{1}{12}(x-1)^4$$

$$\Longrightarrow f(1) = 3 \quad f'(1) = 0 \quad \frac{f''(1)}{2} = 1 \quad \frac{f'''(1)}{3!} = \frac{1}{12} \quad \frac{f^{(4)}(1)}{4!} = \frac{1}{12}$$

$$\Longrightarrow f(1) = 3 \quad f'(1) = 0 \quad f''(1) = 2 \quad f'''(1) = \frac{3 \cdot 2}{12} = \frac{1}{2} \quad f^{(4)}(1) = \frac{4 \cdot 3 \cdot 2}{12} = 2$$

Long-answer questions: You must show your work, your work must be legible and well-justified, and you must record your answers in the spaces provided.

7. (3 points) Using the definition of the derivative (i.e., from first principles), determine the derivative of $f(x) = \sqrt{x-5}$. Make sure you start by writing the definition and use proper mathematical notation throughout.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)-5} - \sqrt{x-5}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h-5} - \sqrt{x-5}}{h}\right)\left(\frac{\sqrt{x+h-5} + \sqrt{x-5}}{\sqrt{x+h-5} + \sqrt{x-5}}\right)$$

$$= \lim_{h \to 0} \frac{(x+h-5) - (x-5)}{h\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-5} + \sqrt{x-5}}$$

$$= \frac{1}{\sqrt{x+0-5} + \sqrt{x-5}}$$

$$= \frac{1}{2\sqrt{x-5}}$$

8. (2 points) Using methods from calculus and algebra, evaluate the following limit if it exists; otherwise, justify mathematically why it does not exist. **Make sure you identify any indeterminate forms you encounter and any theorems that you use from class.**

$$\lim_{x\to 0} \frac{e^x \sin(3x) - 3x}{3x^2} \leftarrow \text{indeterminate form of type } \frac{0}{0}$$

*l'Hopital's Rule

$$= \lim_{x\to 0} \frac{e^x \sin(3x) + e^x(\cos(3x))(3) - 3}{6x}$$

$$= \lim_{x\to 0} \frac{e^x(\sin(3x) + 3\cos(3x)) - 3}{6x} \leftarrow \text{indet. form of type } \frac{0}{0}$$

*l'Hopital's Rule

$$= \lim_{x\to 0} \frac{e^x(\sin(3x)+3\cos(3x)) + e^x(3\cos(3x) - 9\sin(3x)) - 0}{6}$$

$$= \lim_{x\to 0} \frac{e^x(-8\sin(3x) + 6\cos(3x))}{6} = \frac{6}{6} = 1$$

* provided the new limit exists or is $\pm\infty$

9. (3 points) Let $f(x) = 2x^4 - 16x^2 + 7$ for $-3 \le x \le 3$. Find the absolute maximum and minimum of f on this interval, and the values of x at which f attains these extreme values.

Absolute maximum value $\boxed{25}$ attained at the point(s) $x =$ $\boxed{\pm 3}$

Absolute minimum value $\boxed{-25}$ attained at the point(s) $x =$ $\boxed{\pm 2}$

Justify your work below!

$$f'(x) = 8x^3 - 32x$$
$$0 = 8x^3 - 32x$$
$$0 = 8x(x^2 - 4)$$
$$x = 0 \qquad x = \pm 2$$

$$f(0) = 7$$
$$f(-2) = 2(-2)^4 - 16(-2)^2 + 7 = -25$$
$$f(2) = -25$$

ABS. MIN.

$$f(-3) = 25$$
$$f(3) = 25$$

ABS. MAX

10. (6 points) Our goal for this question is to produce the graph of $f(x) = \dfrac{-x}{x^3-1}$.

We provide you with the 1st and 2nd derivatives of f below, which may be useful:

$$f'(x) = \frac{1+2x^3}{(x^3-1)^2} \quad \text{and} \quad f''(x) = \frac{-6x^2(x^3+2)}{(x^3-1)^3}$$

(a) Write the domain of f. $\boxed{\{x \in \mathbb{R} : x \neq 1\}}$

(b)Determine the critical number(s) of f (round your answers to 3 decimal places), or write 'none' if there aren't any.

$$0 = \frac{1+2x^3}{(x^3-1)^2} \Rightarrow 0 = 1+2x^3 \Rightarrow x^3 = -\tfrac{1}{2} \Rightarrow x = \sqrt[3]{-\tfrac{1}{2}} \approx -0.794$$

Critical number(s) of f: $\boxed{-0.794}$

(c) Determine the intervals where f is increasing/decreasing. Identify any local minimum/maximum points. Justify your answers using the first derivative, and write your findings in a clearly organized table.

VA $x=1$

	$(-\infty, \sqrt[3]{-\frac{1}{2}})$	$(\sqrt[3]{-\frac{1}{2}}, 1)$	$(1, \infty)$
Sign of f'	−	+	+
behaviour of f	DEC	INC	INC

f has a local min. @ $x = \sqrt[3]{-\frac{1}{2}}$

(d)Determine the inflection point candidate(s) of f (round your answers to 3 decimal places), or write 'none' if there aren't any.

$$0 = \frac{-6x^2(x^3+2)}{(x^3-1)^3} \Rightarrow 0 = -6x^2(x^3+2)$$

$x = 0$, $x = \sqrt[3]{-2} \approx -1.260$

IP candidate(s) of f: $\boxed{0 \text{ and } -1.260}$

(e) Determine the intervals where f is concave up/down. Identify any inflection points. Justify your answers using the second derivative, and write your findings in a clearly organized table.

	$(-\infty, \sqrt[3]{-2})$	$(\sqrt[3]{-2}, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f''	−	+	+	−
behaviour of f	CONCAVE DOWN	CONCAVE UP	CONCAVE UP	CONCAVE DOWN

IP @ $x = \sqrt[3]{-2}$

(f) Recall that $f(x) = \frac{-x}{x^3-1}$. Determine all vertical and/or horizontal asymptotes of f, if any. Show your work.

$$\lim_{x\to\infty} \frac{-x}{x^3-1} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(-\frac{1}{x^2}\right)}{\cancel{x^3}\left(1-\frac{1}{x^3}\right)} = \lim_{x\to\infty} \frac{\left(-\frac{1}{x^2}\right)}{1-\frac{1}{x^3}} = \frac{0}{1-0} = 0$$

$$\lim_{x\to-\infty} \frac{-x}{x^3-1} = 0$$

$\therefore$ f has H.A. $y=0$ as $x\to\infty$ and as $x\to-\infty$

$$\lim_{x\to 1^-} \frac{-x}{x^3-1} \begin{matrix} \rightsquigarrow -1 \\ \rightsquigarrow 0^- \end{matrix} = \infty$$

$$\lim_{x\to 1^+} \frac{-x}{x^3-1} \begin{matrix} \rightsquigarrow -1 \\ \rightsquigarrow 0^+ \end{matrix} = -\infty$$

$\therefore$ f has a V.A. @ $x = 1$.

(g) Using your answers to (a)–(f), and any further values or limits you need, sketch the graph of $y = f(x)$ on the grid given below, and label all important points found in parts (a)–(f).