Solution to Test 1 (version C)

MAT1320B, Fall 2019
Total = 20 marks

Part I. Multiple-Choice Questions (6 × 2 = 12 marks)

AEBCDF

1. Assume some values of a one-to-one function $y = f(x)$ are given in the following table

x	1	2	3	4	5
$f(x)$	4	5	1	2	3
$g(x)$	2	4	5	3	1

Which one of the following statements is true?

(A) $(f \circ g)(2) = 2, f^{-1}(2) = 4$.
(B) $(f \circ g)(2) = 1, f^{-1}(2) = 5$.
(C) $(f \circ g)(2) = 2, f^{-1}(2) = 1$.
(D) $(f \circ g)(2) = 1, f^{-1}(2) = 1$.
(E) $(f \circ g)(2) = 2, f^{-1}(2) = 5$.
(F) $(f \circ g)(2) = 1, f^{-1}(2) = 4$.

Solution. (A) $(f \circ g)(2) = f(g(2)) = f(4) = 2$. Since $f(4) = 2$, $f^{-1}(2) = 4$.

2. Let $f(x) = \begin{cases} ax+4, & x \le 2, \\ ax^2 - 8, & x > 2. \end{cases}$ For which value of a is $f(x)$ continuous for all real numbers?

(A) 9; (B) 2; (C) 4; (D) 1; (E) 6; (F) 8.

Solution. (E) This function is continuous when $x < 2$ and when $x > 2$. When $x = 2$,

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(ax+4) = 2a+4$, and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(ax^2-8) = 4a-8$. Hence, $2a + 4 = 4a - 8$, $2a = 12$, $a = 6$.

3. If $3^x = 2^{x+1}$, then $x =$

(A) $\dfrac{\ln(3/2)}{\ln 3}$; (B) $\dfrac{\ln 2}{\ln(3/2)}$; (C) $\ln\dfrac{3}{2}$;
(D) $\ln\dfrac{4}{3}$; (E) $\dfrac{\ln(3/2)}{\ln 2}$; (F) $\dfrac{\ln 3}{\ln(3/2)}$.

Solution. (B) Take the natural logarithm on both sides of the equation: $\ln 3^x = \ln 2^{x+1}$.

By the property of the logarithm, $x \ln 3 = (x + 1) \ln 2$. $x(\ln 3 - \ln 2) = \ln 2$.
$x = \dfrac{\ln 2}{\ln 3 - \ln 2} = \dfrac{\ln 2}{\ln(3/2)}$.

4. Find the one-side limit $\lim\limits_{x\to -1^-} \dfrac{2x^2 - x - 3}{|x+1|}$.

(A) −1; (B) 1; (C) 5; (D) −5; (E) −3; (F) 3.

Solution. (C) $\lim\limits_{x\to -1^-} \dfrac{2x^2 - x - 3}{|x+1|} = \lim\limits_{x\to -1^-} \dfrac{(2x-3)(x+1)}{-(1+x)} = -\lim\limits_{x\to -1^-} (2x-3) = 5$.

5. If $f(x) = e^{-2x}\sin\left(\dfrac{\pi}{2}x\right)$, then the equation of the tangent line of the graph of $f(x)$ at the point $(1, e^{-2})$ is

(A) $y = e^{-2}x$; (B) $y = 2e^{-2}x - e^{-2}$; (C) $y = -e^{-2}x + 2e^{-2}$;
(D) $y = -2e^{-2}x + 3e^{-2}$; (E) $y = 3e^{-2}x - 2e^{-2}$; (F) $y = -3e^{-2}x + 4e^{-2}$.

Solution. (D) By the chain rule, $\dfrac{d}{dx}e^{-2x} = -2e^{-2x}$, and $\dfrac{d}{dx}\sin\left(\dfrac{\pi}{2}x\right) = \dfrac{\pi}{2}\cos\left(\dfrac{\pi}{2}x\right)$. Hence, by the product rule, $f'(x) = -2e^{-2x}\sin\left(\dfrac{\pi}{2}x\right) + \dfrac{\pi}{2}e^{-2x}\cos\left(\dfrac{\pi}{2}x\right)$. When $x = 1$, $f'(1) = -2e^{-2}$. The equation of the tangent line of the graph of $f(x)$ at the point $(1, e^{-2})$ is $y = -2e^{-2x}(x - 1) + e^{-2}$, or $y = -2e^{-2}x + 3e^{-2}$.

6. Suppose a function $y = f(x)$ is defined implicitly by the equation $\dfrac{x}{y} + 2x + y^2 = 3$ near a point $(2, -1)$. Then the derivative of this function at the point $(2, -1)$ is

(A) $-\dfrac{1}{4}$; (B) 1; (C) $\dfrac{2}{3}$; (D) $\dfrac{1}{2}$; (E) −2; (F) $\dfrac{1}{4}$.

Solution. (F) Taking the derivative on both sides with respect to x, we have

$\dfrac{y - xy'}{y^2} + 2 + 2yy' = 0$. When $x = 2$, and $y = -1$, $-1 - 2y' + 2 - 2y' = 0$. $y' = \dfrac{1}{4}$.

Part II. Detailed Answer Question (8 marks)

1. (4 marks) Consider function $f(x) = \dfrac{3x^2+4x-4}{2x^2+3x-2}$.

Find the horizontal/vertical asymptote(s) of this graph, if any. (Answer NONE if there is no horizontal or vertical asymptote.)

Answer. The horizontal asymptote(s) of the graph of $f(x)$ is / are $\underline{y = 3/2}$.

The vertical asymptote(s) of the graph of $f(x)$ is/ are $\underline{x = 1/2}$.

Justification. $\lim_{x\to\infty}\dfrac{3x^2+4x-4}{2x^2+3x-2} = \lim_{x\to\infty}\dfrac{3+4/x-4/x^2}{2+3/x-2/x^2} = \dfrac{3}{2}$, and $\lim_{x\to-\infty}\dfrac{3x^2+4x-4}{2x^2+3x-2} = \lim_{x\to-\infty}\dfrac{3+4/x-4/x^2}{2+3/x-2/x^2} = \dfrac{3}{2}$. The graph of $f(x)$ has one horizontal asymptote $y = \dfrac{3}{2}$.

Let $2x^2 + 3x - 2 = 0$. $x = -2$, $x = \dfrac{1}{2}$. Since $\lim_{x\to-2}\dfrac{3x^2+4x-4}{2x^2+3x-2} = \lim_{x\to-2}\dfrac{(3x-2)(x+2)}{(2x-1)(x+2)} = \lim_{x\to-2}\dfrac{3x-2}{2x-1} = \dfrac{8}{5}$, $x = -2$, is not a vertical asymptote. Since the numerator is not zero at $x = \dfrac{1}{2}$, the graph of $f(x)$ has only one vertical asymptotes $x = \dfrac{1}{2}$.

2. (4 marks) (a) (1 mark) Give the definition of the derivative of a function $y = f(x)$ at a point $x = a$.

(b) (3 marks) **Use the definition of the derivative** to find the derivative of the function $y = \sqrt{4x-3}$ at $x = 1$.

Solution. (a) $f'(a) = \lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$.

(b) $y'(1) = \lim_{h\to 0}\dfrac{1}{h}\left(\sqrt{4(1+h)-3}-1\right) = \lim_{h\to 0}\dfrac{(\sqrt{1+4h}-1)(\sqrt{1+4h}+1)}{h(\sqrt{1+4h}+1)} = \lim_{h\to 0}\dfrac{4}{\sqrt{1+4h}+1} = 2$.