Test 4 MATH 1004D, Fall, 2019

Last name: First name: Student no.:

This test has two parts, the first part has 4 multiple choices questions (3 marks each) and the second part has 4 long answers. There are **3 bonus marks**. The final maximum mark is 30 Calculator is NOT allowed

A1: Let $g(x) = \int_1^x \sqrt{1+8t^2}\,dt$, then $g'(1)$ is

a) 0 b) 3 c) 2 d) $\sqrt{3}$ (e) None.

Ans (b)

A2: What is $\frac{d}{dx}\left(\int_1^{x^2} \frac{e^t}{t}\,dt\right)$?

(a) $\frac{e^{x^2}}{x^2}$ (b) $2\frac{e^{x^2}}{x}$ (c) $\frac{2e^{x^2}}{x^2}$ (d) $x^2e^{x^3}$ (e) None.

Ans (b)

A3: The value of $\int_{e^2}^{e^5} \frac{1}{x}\,dx$ is

(a) 3 (b) 4 (c) e^3 (d) $\ln 3$ (e) None.

Ans (a)

A4: Evaluate $\int_0^1 (\sqrt[3]{x} + \sqrt[4]{x})\,dx$

(a) $\frac{1}{3}+\frac{1}{4}$ (b) 1 (c) $\frac{2}{3}+\frac{3}{4}$ (d) $\frac{3}{4}+\frac{4}{5}$ (e) None.

Ans (d)

B1-[5 marks]: Let $I = \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} \left(1+\left(\frac{i}{n}\right)^2\right)\frac{1}{n}$. Using a Riemann sum, convert I to a definite integral. (You do not need to find the value of the integral).

$$f(x) = (1+x^2)$$
$$x_i^* = x_i = \frac{i}{n}$$
$$\Delta x = \frac{1}{n}$$
$$a = x_0 = \frac{0}{n} = 0,\ b = x_n = \frac{n}{n} = 1$$

Then, $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\left(\frac{i}{n}\right)^2\right)\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}(1+x_i^{*2})\Delta x = \int_0^1 (1+x^2)dx$

B2-[5 marks]: Evaluate $\int_0^1 x\,dx$ using the definition of definite integral only. (Hint: you may use $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$)

$$f(x) = x,\ a = 0,\ b = 1, \text{ then } \Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i = a + i\Delta x = 0 + i(\frac{1}{n}) = \frac{i}{n}$$

$$x_i^* \in [x_{i-1}, x_i] = [\frac{i-1}{n}, \frac{i}{n}]$$

$$\text{put } \ x_i^* = x_i = \frac{i}{n}$$

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} x_i^*\Delta x = \sum_{i=1}^{n} \frac{i}{n}\frac{1}{n} = \sum_{i=1}^{n} \frac{i}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n} i$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \implies \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2} \times \frac{n(n+1)}{2} = \frac{n+1}{2n}.$$

$$\text{Then, } \int_0^1 xdx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$$

B3-[5 marks]: Evaluate $\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx$

$$u = x^2 + 1 \implies du = 2xdx$$

$$x = 0 \implies u = 1$$

$$x = \sqrt{3} \implies u = 4$$

$$\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx = \frac{1}{2}\int_1^4 \sqrt{u}\,du = \left[\frac{1}{2}(\frac{2}{3}u^{\frac{3}{2}})\right]_1^4 = \frac{1}{3}(4^{\frac{3}{2}} - 1^{\frac{3}{2}}) = \frac{1}{3}(8-1) = \frac{7}{3}$$

B4-[4+2 marks]: First evaluate $\int \frac{x}{1+x^4}\,dx$, then find $\int_0^1 \frac{x}{1+x^4}\,dx$.

$$x^4 = (x^2)^2 \implies u = x^2 \implies du = 2xdx$$

$$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{2x\,dx}{1+(x^2)^2} = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1} u = \frac{1}{2}\tan^{-1}(x^2) + C$$

$$\int_0^1 \frac{x}{1+x^4}\,dx = \frac{1}{2}\left[tan^{-1}(x^2)\right]_0^1 = \frac{1}{2}(tan^{-1}(1) - tan^{-1}(0)) = \frac{1}{2}(\frac{\pi}{4} - 0) = \frac{\pi}{8}$$