Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

MAT 1341A – The Midterm Test I (v.1) **Instructor: K. Zaynullin**

Last name: ______________________________

First name: ______________________________

Student number:

Please, read the following instructions carefully:

- You have 80 minutes to complete this test. **Do not detach** the pages of this examination. Read each question carefully. Where it is possible to check your work, do so.
- You can use the backs of the pages and the last page for computations.
- This is a closed book exam, and no notes of any kind are allowed. The use of programmable calculators, cell phones, laptops, pagers or any text storage or communication device is not permitted.

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	6	7	8	Total
Mark									
Out of	1	1	1	1	5	4	1	1	15

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1. Find parametric equations of the line containing the point $(0, 2, 1)$ and which is parallel to two planes $-x + y + 3z = 0$ and $-5x + 3y + 4z = 1$. (1)

`cross (X) the correct answer:`

- [A] $x = 5t,\ y = 2 + 11t,\ z = 1 + 2t$
- [B] $x = 5t,\ y = 2 + 11t,\ z = 1 - 2t$
- [C] $x = -5t,\ y = -2 + 5t,\ z = 1 - 10t$
- [D] $x = 0,\ y = -2t,\ z = t$
- [E] $x = -3t,\ y = -2 + 11t,\ z = 1 + 2t$
- [F] $x = 5 - 5t,\ y = 13 - 11t,\ z = 4 - 2t$

Solution: A direction vector for this line is

$$d = (-1, 1, 3) \times (-5, 3, 4) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 3 \\ -5 & 3 & 4 \end{vmatrix} = (-5, -11, 2).$$

Hence [B] is correct

2. Given $u = (3, 1, -2)$, $v = (2, 0, -1)$, $w = (-1, 1, 2)$ find the cosine of the angle between $(v \times w)$ and $(u \times v)$.

`cross (X) the correct answer:`

- $\boxed{\text{A}}$ $\frac{2}{21}$
- $\boxed{\text{B}}$ $-\frac{1}{21}$
- $\boxed{\text{C}}$ $-\frac{1}{\sqrt{21}}$
- $\boxed{\text{D}}$ $\frac{\sqrt{2}}{\sqrt{21}}$
- $\boxed{\text{E}}$ $-\frac{1}{\sqrt{7}}$
- $\boxed{\text{F}}$ $\frac{2}{\sqrt{7}}$

Solution:

$$v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ -1 & 1 & 2 \end{vmatrix} = (1, -3, 2).$$

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 2 & 0 & -1 \end{vmatrix} = (-1, -1, -2).$$

Hence,

$$\frac{(v \times w) \cdot (u \times v)}{||v \times w||||u \times v||} = -\frac{1}{\sqrt{21}}.$$

So the correct answer is $\boxed{\text{C}}$.

3. Mark whether each of the following statements is TRUE or FALSE in the respective box. (each correct answer is 1/4pt)

- It is possible that a system of linear equations has exactly 3 solutions.

  ANSWER: [ ] FALSE

- A homogeneous system of linear equations can have infinitely many solutions.

  ANSWER: [ ] TRUE

- There exists a linear system of five equations such that its coefficient matrix has rank 6.

  ANSWER: [ ] FALSE

- If a system has 3 equations and 5 variables, then this system always has infinitely many solutions.

  ANSWER: [ ] FALSE

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4. If the coefficient matrix A in a homogeneous system in 20 variables of 16 equations is known to have rank 9, how many parameters are there in the general solution? (1)

`cross (X) the correct answer:`

[A] 11

[B] 10

[C] 6

[D] 21

[E] 17

[F] 4

Solution: Homogeneous system is consistent. So the number of parameters is 20 - 9=11. So the correct answer is [A].

5. Suppose $e, f \in \mathbb{R}$ and consider the linear system in x, y and z:

$$\begin{cases} 2x - 2y + ez & = f \\ 2x + y + z & = 0 \\ x + z & = -1 \end{cases}$$

5(a) If $(A \mid b)$ is the augmented matrix of the system above, find the rank of A and the rank of $(A \mid b)$ for **all** values of e and f. (2)

(justify your answers)

Solution: We have

$$(A \mid b) = \begin{pmatrix} 2 & -2 & e & \mid & f \\ 2 & 1 & 1 & \mid & 0 \\ 1 & 0 & 1 & \mid & -1 \end{pmatrix} \xrightarrow{Gauss\ elimination} \begin{pmatrix} 1 & 0 & 1 & \mid & -1 \\ 0 & 1 & -1 & \mid & 2 \\ 0 & -2 & e-2 & \mid & f+2 \end{pmatrix}$$

$$\xrightarrow{Gauss\ elimination} \begin{pmatrix} 1 & 0 & 1 & \mid & -1 \\ 0 & 1 & -1 & \mid & 2 \\ 0 & 0 & e-4 & \mid & f+6 \end{pmatrix}$$

Hence,

$$rank\ A = \begin{cases} 2 & if\ e = 4 \\ 3 & if\ e \neq 4 \end{cases}$$

$$rank\,(A \mid b) = \begin{cases} 2 & if\ e = 4\ and\ f = -6 \\ 3 & otherwise \end{cases}$$

5(b) Using part (a), find all values of e and f so that this system has

(i) a unique solution (1)

Solution: it has a unique solution $\iff rank\, A = rank\,(A \mid b) = 3 \iff e \neq 4$.

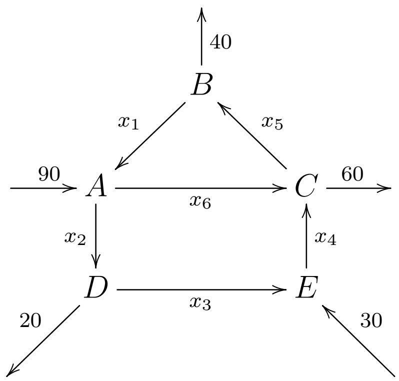
(ii) infinitely many solutions (1)

Solution: it has infinitely many solutions $\iff rank\, A = rank\,(A \mid b) < 3 \iff e = 4$ and $f = -6$.

(iii) no solutions (1)

Solution: it has no solutions $\iff rank\, A < rank\,(A \mid b) \iff e = 4$ and $f \neq -6$.

6. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passes along the indicated streets during the same period.



6(a) Write down a system of linear equations which describes the traffic flow **together with all the constrains on the variables** x_i, $i = 1, \ldots, 6$. (1)

(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this)

Solution:

Intersection	Flow in = Flow out
A	$90 + x_1 = x_2 + x_6$
B	$x_5 = x_1 + 40$
C	$x_4 + x_6 = x_5 + 60$
D	$x_2 = x_3 + 20$
E	$30 + x_3 = x_4.$

Oneway streets give constrains $x_i \geq 0$, $i = 1, \ldots, 6$. Since each x_i is a number of cars, x_i has to be an integer.

6(b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 50 \\ 0 & 0 & 1 & 0 & -1 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Give the general solution of this system (1)

(Ignore the constrains from (a) at this point)

Solution: Set $x_5 = s$ and $x_6 = t$ to be parameters. Then

$$\begin{aligned} x_1 &= -40 + s \\ x_2 &= 50 + s - t \\ x_3 &= 30 + s - t \\ x_4 &= 60 + s - t \\ x_5 &= s \\ x_6 &= t \end{aligned}$$

6(c) If the road ED was closed in the middle due to roadwork, find the minimum flow along the road AC **using your results from (b)** (2)

(you must justify all your answers: correct answer without justification is 1pt only)

Solution: ED is closed $\iff x_3 = 0 \iff s - t = -30$. Then assuming the constrains $x_i \geq 0$ we obtain the system of inequalities

$$\begin{cases} x_1 & = -40 + s \geq 0 \iff s \geq 40 \\ x_2 & = 50 - 30 = 20 \\ x_3 & = 0 \\ x_4 & = 60 - 30 = 30 \\ x_5 & = s \geq 0 \\ x_6 & = s - 30 \geq 0 \iff s \geq 30 \end{cases}$$

The flow along AC is $x_6 = s + 30$ so that $x_6 \geq 70$. The minimum flow along x_6 is 70.

7. Mark whether each of the following statements is TRUE or FALSE in the respective box. (each correct answer is 1/4pt)

- For any two 2×2 matrices A and B, we have $(AB)^2 = A^2B^2$.

  ANSWER: [ ] FALSE

- Multiplying a 3×2-matrix A by a 2×3-matrix B one gets a 2×2-matrix AB.

  ANSWER: [ ] FALSE

- There exist two non-zero matrices A and B such that $A + B$ is the zero-matrix.

  ANSWER: [ ] TRUE

- For any two matrices A and B we have $AB = BA$.

  ANSWER: [ ] FALSE

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8. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, and B is a 3×5 matrix,

then the second row of the matrix $A \cdot B$ is (1)

`cross (X) the correct answer:`

[A] the same as the first row of B

[B] the sum of the first and the second rows of B

[C] the sum of the first, the second and the third rows of B

[D] the sum of the first and the third rows of B

[E] the same as the second row of B

[F] the sum of the second and the third rows of B

Solution: Write $B = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$ in block form, where r_i is the i-th row of B. Then

$$A \cdot B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_1 \\ r_1 + r_2 + r_3 \end{pmatrix}$$

So the correct answer is [A].

The last page (use it for computations)