Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

CALCULUS II | Instructor: ELIZABETH MALTAIS

MAT1322E – Test 3 – Monday, March 25, 2019

- Clearly write your name and student number on this test, and **sign it** below to confirm that you have read, understood and agreed to follow these **instructions**:
- This is a 75-minute **closed-book** test. No notes. No calculators.
- The exam consists of 6 questions on 7 pages (including this cover page).
- Each question is worth 2 points.
- maximum points possible = 12 points.
- Read all questions carefully and be sure to follow the instructions for the individual problems.
- All questions are *__long-answer. To receive full marks, your solution must be correct, complete, and show all relevant details.__*

  *Some pages include a multiple-choice question. You will **not** earn any points for circling the correct response without having properly justified your choice.
- You must use **proper mathematical notation and terminology**. Make sure that your notation is consistent with the notation used in class.
- For additional work space, you may use the backs of pages.
  Do not use any of your own scrap paper.
  Page 7 is for scrap work – you may detach Page 7

  Cellular phones, unauthorized electronic devices or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

  † **By signing below, you acknowledge that you have read, understood, and will comply with the above instructions.**

FAMILY NAME:	STUDENT NUMBER:
FIRST NAME:	SIGNATURE:

SOLUTIONS

Question	Q1	Q2	Q3	Q4	Q5	Q6	Total
Maximum points	2 pts	2 pts	2 pts	2 pts	2 pts	2 pts	12 points
Marks obtained							

1. Consider the power series $\sum_{n=0}^{\infty} \frac{(2x+6)^n}{4^n\sqrt{n+1}}$ $\leftarrow a_n = \frac{(2x+6)^n}{4^n\sqrt{n+1}}$

Find its radius and interval of convergence. Show your work!

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x+6)^{n+1}}{4^{n+1}\cdot\sqrt{n+2}} \Big/ \frac{(2x+6)^n}{4^n\sqrt{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2x+6)\cdot\sqrt{n+1}}{4\cdot\sqrt{n+2}}\right|$$

$$= \lim_{n\to\infty}\frac{\sqrt{n+1}}{4\sqrt{n+2}}|2x+6| = \frac{1}{4}|2x+6|$$

$\Rightarrow$ converges when $\frac{1}{4}|2x+6| < 1 \iff |2x+6| < 4$

$\iff -4 < 2x+6 < 4$

$\iff -10 < 2x < -2$

$\iff -5 < x < -1$

length of interval is $-1-(-5) = 4 \Rightarrow R = \frac{4}{2} = 2$

Check convergence at endpoints:

when $x=-5$, we get $\sum_{n=0}^{\infty}\frac{(2(-5)+6)^n}{4^n\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{(-4)^n}{4^n\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ $\leftarrow$ this is an alternating series with $b_n = \frac{1}{\sqrt{n+1}}$

$b_{n+1} = \frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}} = b_n$ for all $n \geq 0$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = 0$ $\therefore$ converges when $x=-4$ by A.S.T.

when $x=-1$, we get $\sum_{n=0}^{\infty}\frac{(2(-1)+6)^n}{4^n\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$ $\leftarrow$ $0 \leq \frac{1}{\sqrt{n+n}} \leq \frac{1}{\sqrt{n+1}}$ (for all $n \geq 1$) and $\frac{1}{\sqrt{2n}} = \frac{1}{\sqrt{2}}\cdot\frac{1}{n^{1/2}}$

since $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ is a p-series with $p=\frac{1}{2}<1$, it is divergent $\therefore \sum\frac{1}{\sqrt{n+1}}$ is divergent by comparison

Radius of convergence $R = \boxed{2}$ Interval of convergence: $\boxed{-5 \leq x < -1}$

or $[-5, -1)$

2a. Find the power series representation for the following function:

$$f(x) = \frac{1}{3+x^3}$$

Show your work!

Using $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$

$$\frac{1}{3+x^3} = \frac{1}{3(1+\frac{x^3}{3})} = \frac{1}{3}\left(\frac{1}{1-(-\frac{x^3}{3})}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-x^3}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{3^{n+1}}$$

$$\text{So } \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{3^{n+1}} = \frac{(-1)^0 x^0}{3^1} + \frac{(-1)^1 x^3}{3^2} + \frac{(-1)^2 x^6}{3^3} + \frac{(-1)^3 x^9}{3^4}$$

Note $f(x) = (3+x^3)^{-1}$
so $f'(x) = -(3+x^3)^{-2}(3x^2)$

Write its first 3 nonzero terms in the box:

$f(x) = \boxed{\frac{1}{3} - \frac{x^3}{9} + \frac{x^6}{27}} + \ldots$

2b. Using your answer from part (a), find the power series representation for $g(x) = \frac{-3x^2}{(3+x^3)^2}$

Notice that $g(x) = f'(x)$

$$\text{Thus } g(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{3^{n+1}}\right] = \frac{d}{dx}\left[\frac{1}{3} - \frac{x^3}{9} + \frac{x^6}{27} - \frac{x^9}{81} + \cdots\right]$$

$$= \sum_{n=0}^{\infty}\frac{3n(-1)^n x^{3n-1}}{3^{n+1}}$$

$$= \sum_{n=0}^{\infty}\frac{n(-1)^n x^{3n-1}}{3^n}$$

when n=0, term is zero, so series can start from n=1

$$= \sum_{n=1}^{\infty}\frac{n(-1)^n x^{3n-1}}{3^n} = \frac{-x^2}{3} + \frac{2x^5}{9} - \cdots$$

Write its first 2 nonzero terms in the box:

$g(x) = \boxed{-\frac{x^3}{3} + \frac{2x^5}{9}} + \ldots$

3a. Find the first four nonzero terms of the Maclaurin series for the function

$$f(x) = \sqrt{1+2x}$$

Show your work!

Using Binomial series $(1+y)^k = \sum_{n=0}^{\infty} \binom{k}{n} y^n$ with $y=2x$ and $k=\frac{1}{2}$

$$f(x) = (1+2x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (2x)^n = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} 2^n x^n$$

$$= \binom{\frac{1}{2}}{0} 2^0 x^0 + \binom{\frac{1}{2}}{1} 2^1 x^1 + \binom{\frac{1}{2}}{2} 2^2 \cdot x^2 + \binom{\frac{1}{2}}{3} 2^3 \cdot x^3 + \ldots$$

$$= 1 \cdot 1 \cdot 1 + \left(\tfrac{1}{2}\right)(2) \cdot x + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{(2)(1)} \cdot 2^2 \cdot x^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) \cdot 2^3 x^3}{(3)(2)(1)}$$

$$= 1 + x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) \cdot 2^2 x^2}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \cdot 2^3 \cdot x^3}{(3)\cancel{(2)}(1)} + \ldots$$

$$= 1 + x + \frac{\cancel{\left(\frac{1}{2}\right)^2}(-1) \cdot \cancel{2^2}}{2} x^2 + \frac{\cancel{\left(\frac{1}{2}\right)^3}(-1)^2(1)\cancel{(3)} \cdot \cancel{2^3}}{\cancel{(3)}(2)(1)} x^3 + \ldots$$

Write its first 4 nonzero terms in the box:

$$f(x) = \boxed{1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 - \ldots} + \ldots$$

3b. Using the Maclaurin series you found in part (a) evaluate the follwoing limit:

$$L = \lim_{x \to 0} \frac{\sqrt{1+2x} - 1 - x}{x^2}$$

Show your work! Do not use l'Hospital's Rule!

$$= \lim_{x \to 0} \frac{\left(\cancel{1+x} - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \ldots\right) \cancel{-1-x}}{x^2}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{2}x^2 + \frac{1}{2}x^3 + \ldots \text{(other terms with powers } x^n, n>3)}{x^2}$$

$$= \lim_{x \to 0} -\tfrac{1}{2} + \tfrac{1}{2}x + \ldots \text{(other terms with powers } x^{n-2},\ n>3)$$

$$= -\tfrac{1}{2} + 0 + 0 + \ldots$$

Answer: $L = \boxed{-\frac{1}{2}}$

4. What is the coefficient of x^5 in the power series of the integral

$$\int \sum_{n=1}^{\infty} \frac{x^{2n}}{n}\, dx$$

Show your work!

$\int \sum_{n=1}^{\infty} \frac{x^{2n}}{n}\, dx = C + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)(n)}$ ← for x^5, we need $2n+1=5 \Rightarrow n=2$

$\therefore$ the coefficient of x^5 is $\frac{1}{(2(2)+1)(2)} = \frac{1}{5\cdot 2} = \frac{1}{10}$

The coefficient of x^5 is... Circle the best response:

A. $\frac{1}{15}$ **B.** $\frac{1}{5}$ **C.** 0 **D.** $\frac{1}{10}$ **E.** 15 **F.** $\frac{2}{5}$

G. none of the above

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5. For which value(s) of k is $y(t) = e^{kt}$ a solution to the differential equation $y'' + 5y' - 6y = 0$? Show your work!

$y(t) = e^{kt} \Rightarrow y'(t) = ke^{kt} \Rightarrow y''(t) = k^2 e^{kt}$

For which values of k is $y'' + 5y' - 6y = 0$?

when $(k^2 e^{kt}) + 5(ke^{kt}) - 6(e^{kt}) = 0$

$\Longleftrightarrow e^{kt}(k^2 + 5k - 6) = 0$

$\Longleftrightarrow e^{kt}(k+6)(k-1) = 0$

never zero $\quad k=-6 \quad k=1$

Circle the best response:

A. $k = -5$ and $k = -1$ **B.** $k = -6$ and $k = 1$ **C.** $k = \ln(3)$ and $k = 1$

D. $k = -3$ and $k = -2$ **E.** $k = 2$ and $k = 3$ **F.** $k = \ln(3)$ and $k = e$

G. none of the above

6a. Find the general solution of the following separable differential equation:

$$\frac{dy}{dx} = 1 + y + x^2 + x^2 y$$

Show your work!

$$\Rightarrow \frac{dy}{dx} = 1+y+x^2(1+y) = (1+y)(1+x^2)$$

$$\Rightarrow \int \frac{1}{1+y}\,dy = \int (1+x^2)\,dx$$

$$\Rightarrow \ln|1+y| = x+\frac{x^3}{3} + C$$

$$\Rightarrow |1+y| = e^{x+\frac{x^3}{3}+C} = e^C \cdot e^{x+\frac{x^3}{3}}$$

$$\Rightarrow 1+y = \pm e^C \cdot e^{x+\frac{x^3}{3}}$$

$$\Rightarrow y = \pm e^C e^{x+\frac{x^3}{3}} - 1$$

$$\Rightarrow y = Ae^{x+\frac{x^3}{3}} - 1 \quad \text{(where } A = \pm e^C \text{ is some constant)}$$

The general solution is $y(x) = \boxed{Ae^{x+\frac{x^3}{3}} - 1}$

6b. Now find the particular solution to the above differential equation with the initial condition

$$y(0) = -3$$

$$-3 = y(0) = Ae^{0+\frac{0^3}{3}} - 1$$

$$\Rightarrow -3 = A - 1$$

$$\Rightarrow A = -2$$

The particular solution is $y(x) = \boxed{-2e^{x+\frac{x^3}{3}} - 1}$