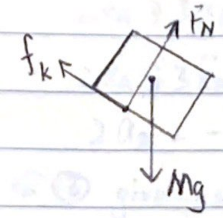
MCG 2108

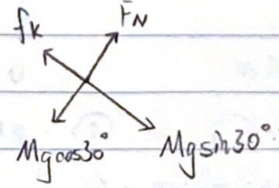
Yuming

HW 2

1. FBDs:

Frame & sphere:





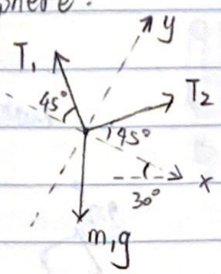
$Mg = (m_1 + m_2)g$

$F_N = Mg\cos30^\circ$

$f_k = \mu F_N = \mu Mg\cos30^\circ$

$$a = \frac{Mg\sin30^\circ - f_k}{M} = g\sin30^\circ - \mu g\cos30^\circ$$

Sphere:



$\Sigma F_y = -m_1 g\cos30^\circ + T_1\sin45^\circ + T_2\sin45^\circ = 0$

$\Sigma F_x = m_1 g\sin30^\circ + T_2\cos45^\circ - T_1\cos45^\circ = a m_1$

$$\therefore \begin{cases} \frac{\sqrt{2}}{2}T_1 + \frac{\sqrt{2}}{2}T_2 = m_1 g \cdot \frac{\sqrt{3}}{2} \\ \frac{\sqrt{2}}{2}T_2 - \frac{\sqrt{2}}{2}T_1 = \cancel{m_1 g\sin30^\circ} - m_1 g \mu \frac{\sqrt{3}}{2} \quad \cancel{-m_1 g\sin30^\circ} \end{cases}$$

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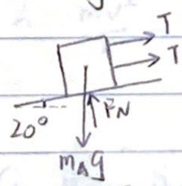
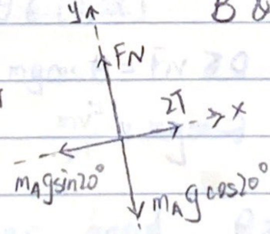
Solve for T_1, T_2,

$$\begin{cases} T_1 = 72.09\,N \ \angle 75^\circ \\ T_2 = 48.06\,N \ \angle 15^\circ \end{cases}$$ $\boxed{\text{ANS.}}$

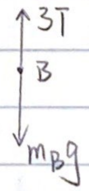
2. FBDs: (Tension along the wire is uniform)

A & pully:





B & pully:



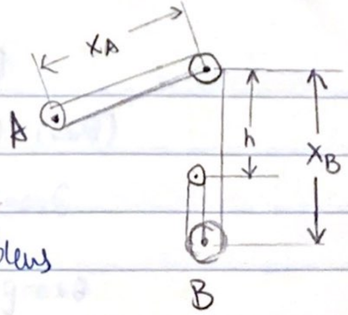
$\Sigma F_y = F_N - m_A g\cos20^\circ$

$\Sigma F_x = 2T - m_A g\sin20^\circ = a_A m_A$ ①

$3T - m_B g = -a_B m_B$ ②

X ok

Wire length $= 2x_A + 3x_B - 2h = $ Constant



$\therefore \quad 2\dot{x}_A + 3\dot{x}_B = 0$

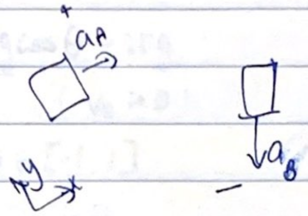
$2\ddot{x}_A + 3\ddot{x}_B = 0$

$2a_A = -3a_B$ ③

~~in reverse directions~~

①, ②, ③ give us : $\begin{cases} 3T = m_B(a_B + g) \\ 2T = m_A(a_A + g\sin 20°) \\ 2a_A = -3a_B \end{cases}$

OK



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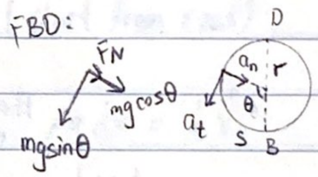
$\Rightarrow \begin{cases} a_A = -1.0236 \ m/s^2 \\ a_B = 0.6824 \ m/s^2 \end{cases}$

Thus, $\vec{a_A} = 1.0236 \ m/s^2 \ \bar{\nabla} 20°$ $\boxed{\text{ANS.}}$

$\vec{a_B} = 0.6824 m/s^2 \uparrow$

3. On the loop : $\vec{a} = \vec{a_n} + \vec{a_t}$

FBD:



$a_t = \dfrac{mg\sin\theta}{m} = g\sin\theta$

$a_n = \dfrac{v^2}{r} = \dfrac{mg\cos\theta + F_N}{m}$

Obviously, $F_N \geq 0$ (positive direction is pointing toward to center of the loop)

$\theta \in [0, 2\pi]$

$\therefore \quad \dfrac{mv^2}{r} - mg\cos\theta = F_N \geq 0$

$\dfrac{mv^2}{r} \geq mg\cos\theta \quad (*)$

On the other hand, $\theta = \frac{s}{r}$, $a_t ds = v dv$

$\therefore a_t ds = g\sin\frac{s}{r} ds = v dv$

Intergrate: $\int_{s_0}^{s} g\sin\frac{s}{r} ds = \int_{v_0}^{v} v dv$

$\left[-rg\cos\frac{s}{r}\right]_{s_0}^{s} = \frac{1}{2}(v^2 - v_0^2)$

$\because s_0 = 0 \quad \therefore rg - rg\cos\frac{s}{r} = \frac{1}{2}(v^2 - v_0^2)$

$$v^2 = v_0^2 + 2rg(1-\cos\theta)$$

Plug in (*): $\frac{m}{r}[v_0^2 + 2rg(1-\cos\theta)] \geq mg\cos\theta$

$$v_0^2 + 2rg(1-\cos\theta) \geq rg\cos\theta$$

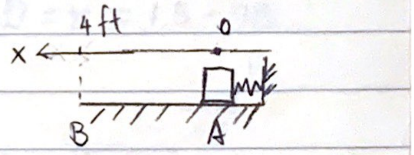
$$v_0^2 \geq 3rg\cos\theta - 2rg$$

$\therefore v_0 \geq \sqrt{|3rg\cos\theta - 2rg|} \quad (v_0 > 0)$

$\because \theta \in [0, 2\pi] \quad \therefore \cos\theta \in [-1, 1]$

$\therefore v_0 \geq \sqrt{5gr}$

On AB:



$\Sigma F = F_s - f_k$

$a = a_s - a_f$ where $a_f = \frac{mg\mu}{m} = \mu g$, $a_s = \begin{cases} \frac{k}{m}(0.25ft - x), & x \in [0, 0.25) \\ 0, & x \geq 0.25\,ft \end{cases}$

$\because a\,dx = v\,dv$

Intergrate: $\int_0^{4ft} a\,dx = \int_0^{v_0} v\,dx$ (start from rest)

10/10 $\int_0^{0.25ft} \frac{k}{m}(0.25 - x)\,dx + \int_{0.25}^{4} 0\,dx - \int_0^{4ft} \mu g\,dx = \frac{1}{2}v_0^2$

$$\left[\frac{k}{m}(0.25x - \frac{1}{2}x^2)\right]_0^{0.25} - [\mu g x]_0^4 = \frac{1}{2}v_0^2$$

$$\frac{(0.25ft)^2}{2} \cdot \frac{k}{m} - (4ft)\mu g = \frac{1}{2}v_0^2$$

$$(0.25ft)^2\frac{k}{m} - (8ft)\mu g = v_0^2 \geq 5gr$$

$$(0.25ft)^2 k \geq 5mgr + (8ft)mg\mu$$

$(W = mg = 5lb, r = 2ft, \mu = 0.3)$ $\quad k \geq mg\frac{5r + (8ft)\mu}{(0.25ft)^2} = 99.2\,lb/ft$

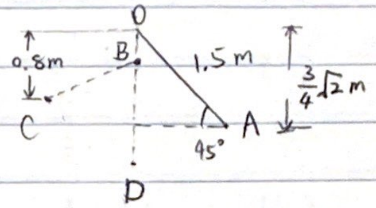
Therefore, the minimum k is 99.2 lb/ft $\boxed{\text{ANS.}}$

4. Only kinetic energy and gravitational potential energy is included.

Conservation of Energy: $E_{p_0} + E_{k_0} = E_p + E_k$

$$E_k = E_{p_0} - E_p + E_{k_0}$$

$$\frac{1}{2} m v^2 = mg\left(0.8 - \frac{3}{4}\sqrt{2}\right) + \frac{1}{2} m v_0^2$$



$$v^2 = 2g\left(0.8 - \frac{3}{4}\sqrt{2}\right) + 3^2 = 3.886\, m^2/s^2$$

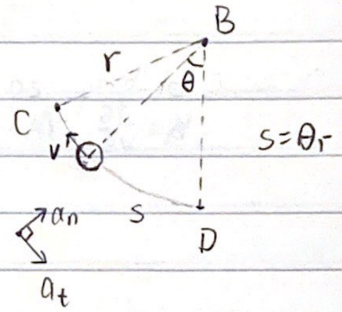
10/10 $v = 1.971\ m/s$ $\boxed{\text{ANS.}}$

OA = OD

r = BD = BC

5. Assume the cord tension is 0 on DC, let BD = r = 1.5 − OB

① at D, (s = 0, θ = 0)



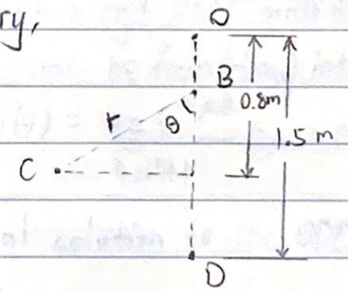
s = θr

Conservation of energy: $E_{kD} = E_{k0} + (E_{p_0} - E_{pD})$

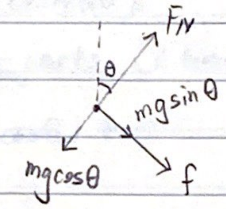
$$\frac{1}{2} m v_D^2 = \frac{1}{2} m v_0^2 + mg\left(1.5 - \frac{3}{4}\sqrt{2}\right)$$

$$\Rightarrow v_D = 4.1976\ m/s$$

② at C, from geometry,



FBD:



$$\cos\theta = \frac{r - 0.7}{r}$$

$$\Rightarrow \theta = \arccos\left(1 - \frac{0.7}{r}\right)$$

$$s = r \cdot \arccos\left(1 - \frac{0.7}{r}\right)$$

$$v_c = 0$$

③ Construct Ordinary Differential Equation.

$$a_n = \frac{v^2}{r} = \frac{F_N - mg\cos\theta}{m} \Rightarrow \frac{F_N}{m} = \frac{v^2}{r} + g\cos\theta$$

$$a_t = g\sin\theta + \frac{\mu F_N}{m} = g\sin\theta + \mu\frac{v^2}{r} + \mu g\cos\theta$$

$\because$ a_n doesn't change the magnitude of v

$\therefore$ $a_t ds = -v dv$

$\because$ $s = \theta r$ $\therefore$ $ds = r d\theta$

$\Rightarrow (rg\sin\theta + \mu v^2 + rg\mu\cos\theta) d\theta = -v dv$

$[rg(\sin\theta + \mu\cos\theta) + \mu v^2] d\theta + v dv = 0$ (Obviously not separable)

It's easy to check it is not exact, but it can be made exact by an intergrating factor $e^{2\mu\theta}$

$\therefore$ $\underbrace{e^{2\mu\theta}[rg(\sin\theta + \mu\cos\theta) + \mu v^2]}_{M} d\theta + \underbrace{e^{2\mu\theta} v}_{N} dv = 0$ (Exact ODE)

④ Solve the ODE.

Let $F(v, \theta) = C$ be the general solution, then $\frac{\partial F}{\partial \theta} = M$, $\frac{\partial F}{\partial v} = N$

$\therefore$ $F(v, \theta) = \int N dv = \frac{1}{2} v^2 e^{2\mu\theta} + h(\theta)$

$\Rightarrow \frac{\partial F}{\partial \theta} = \mu v^2 e^{2\mu\theta} + h'(\theta) = M$

$\therefore$ $h'(\theta) = rg e^{2\mu\theta}(\sin\theta + \mu\cos\theta)$

$\therefore$ $h(\theta) = \int h'(\theta) d\theta = rg\left(\int e^{2\mu\theta}\sin\theta \, d\theta + \mu \int e^{2\mu\theta}\cos\theta \, d\theta\right)$

the intergrals can be found through intergrating by parts (2 times each)

Then we get $h(\theta) = \frac{rg e^{2\mu\theta}}{4\mu^2+1}(3\mu\sin\theta - \cos\theta + 2\mu^2\cos\theta) + C'$

Therefore, the general colution to the ODE is

$$\frac{1}{2} v^2 e^{2\mu\theta} + \frac{rg}{4\mu^2+1} e^{2\mu\theta}(3\mu\sin\theta + 2\mu^2\cos\theta - \cos\theta) = C$$

⑤ Plug in initial values at D to find C,

from ① we know, when $\theta = 0$, $v = 4.1976\, m/s$

$\Rightarrow C = 8.8099 + \frac{2\mu^2 - 1}{4\mu^2+1} rg$

⑥ Plug in values at C

from ② we know,

when $v=0$, $\cos\theta = \frac{r-0.7}{r}$, $\sin\theta = \frac{\sqrt{r^2-(r-0.7)^2}}{r}$

elation between u & BD(r)

$$\therefore \frac{rg}{4\mu^2+1} e^{2\mu\cdot\arccos(1-\frac{0.7}{r})} \cdot \left[\frac{3\mu\sqrt{r^2-(r-0.7)^2}}{r} + \frac{(2\mu^2-1)(r-0.7)}{r}\right] = rg\frac{2\mu^2-1}{4\mu^2+1} + 8.8099$$

Once we know r (the length of BD), we can solve μ.

⑦ If we can assume $OB = 0.8$ m, (it's not clear in the question)

then ⑥ becomes

$$\frac{0.8\times 9.81}{4\mu^2+1} \cdot e^{\pi\mu} \cdot (3\mu) = 8.8099 + \frac{2\mu^2-1}{4\mu^2+1} \times 0.8 \times 9.81$$

$$3\mu \cdot e^{\pi\mu} = 1.1226(4\mu^2+1) + (2\mu^2-1)$$

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$$3\mu \cdot e^{\pi\mu} = 6.4903\mu^2 + 2.1226$$

$$e^{\pi\mu} = 2.1634\mu + 0.7075\mu^{-1}$$

Using calculator, $\boxed{\mu = 0.3329}$ $\boxed{\text{ANS.}}$

6. $\because$ the tension is central force

$\therefore$ angular momentum $\vec{H} = \vec{r} \times m\vec{v} = \text{Constant}$

From the equation of path, we can know $OA = 1.5$ m, $OB = 1.2$ m

10 $\therefore \vec{r_A} \times m\vec{v_A} = \vec{r_B} \times m\vec{v_B}$

$OA \cdot v_A \cdot \sin\frac{\pi}{2} = OB \cdot v_B \cdot \sin\frac{\pi}{2}$

$v_B = 2.5$ m/s $\leftarrow$

$\because T = \frac{mv^2}{r}$, the radius of curvature at B, r_B is $\frac{1.5^2}{1.2} = 1.875$ m

(geometry of ellipse)

$$T_B = \frac{0.7\text{kg} \times (2.5\text{m/s})^2}{1.875\text{m}} = 2.333\text{ N} \downarrow \ \boxed{\text{ANS.}}$$