MAT 2384B-Fall 2019-Assignment #1

To be submitted in class on Friday, September 27 at 10:00 AM

Family Name Solution

Given Name ____________________

Student Number ____________________

- **Please print the assignment and write solutions in the space provided.**
- You can use the back of the pages or additional pages if necessary but be sure to indicate it clearly.
- There are six questions in this assignment.
- You have to answer all the questions.
- Please write your arguments in a clear and readable manner.

Question 1. **[6 points]** Solve each of the following IVPs.

1. $\frac{y}{\cos x}y' = x(1+y^2)$, $y(0)=0$
2. $2x^2+y^2+xyy'=0$, $y(1)=1$
3. $(y^3+3x^2\cos y)dx+(3xy^2-x^3\sin y)dy=0$, $y(1)=0$

1) $\frac{y}{\cos x}\frac{dy}{dx} = x(1+y^2) \Rightarrow \frac{y}{1+y^2}dy = x\cos x\,dx$: Seperable ODE

$\int \frac{y}{1+y^2}dy \overset{(*)}{=} \int x\cos x\,dx$. For $\int \frac{y}{1+y^2}dy$, let $u = 1+y^2 \Rightarrow \frac{du}{dy} = 2y \Rightarrow$

$y\,dy = \frac{1}{2}du$ and so $\int \frac{y}{1+y^2}dy = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(1+y^2)$.

For $\int x\cos x\,dx$, we proceed by parts with $u=x$, $v'=\cos x \Rightarrow$

$u'=1$, $v=\sin x$: $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x$.

So $(*)$ gives: $\frac{1}{2}\ln(1+y^2) = x\sin x + \cos x + C$.

$y(0)=0 \Rightarrow \frac{1}{2}\ln 1 = 1 + C \Rightarrow C=-1$: $\frac{1}{2}\ln(1+y^2) = x\sin x + \cos x - 1$

or $\ln(1+y^2) = 2x\sin x + 2\cos x - 2$: implicit solution.

2) $y' = \frac{dy}{dx} \Rightarrow \underbrace{(2x^2+y^2)}_{M}dx + \underbrace{xy}_{N}dy = 0$: ODE with Homogeneous

Coefficients as both $M(x,y) = 2x^2+y^2$ and $N(x,y) = xy$ are homogeneous

of degree 2. let $u = \frac{y}{x} \Rightarrow y = xu$, $\frac{dy}{dx} = u + x\frac{du}{dx} \Rightarrow$

$dy = u\,dx + x\,du$ and the ODE becomes:

$(2x^2+x^2u^2)dx + x^2u(x\,du + u\,dx) = 0 \Rightarrow (2+u^2)dx + (xu\,du + u^2dx) = 0$

$\Rightarrow 2(1+u^2)dx + xu\,du = 0 \Rightarrow \frac{1}{x}dx = -\frac{u}{2(1+u^2)}du$: seperable ODE.

$$\int \frac{1}{x}\,dx = -\int \frac{u}{2(1+u^2)}\,du \Rightarrow \ln x = -\frac{1}{4}\ln(1+u^2) + C$$

$$y(1)=1 \Rightarrow u(1) = \frac{y(1)}{1} = 1 \;;\quad \ln 1 = -\frac{1}{4}\ln 2 + C \Rightarrow C = \frac{1}{4}\ln 2$$

$$\ln x = -\frac{1}{4}\ln(1+u^2) + \frac{1}{4}\ln 2 = \frac{1}{4}\ln\left(\frac{2}{1+u^2}\right) = \ln\left(\frac{2}{1+u^2}\right)^{1/4} \Rightarrow$$

$$x^4 = \frac{2}{1+u^2} \Rightarrow 1+u^2 = \frac{2}{x^4}, \quad u^2 = \frac{2}{x^4} - 1 \Rightarrow u = \sqrt{\frac{2-x^4}{x^4}}$$

(positive since $u(1) = 1 > 0$) $= \dfrac{\sqrt{2-x^4}}{x^2}$.

$$y = xu = \frac{\sqrt{2-x^4}}{x}$$

<u>Verification</u> (Not required)

$$y = \frac{\sqrt{2-x^4}}{x} \Rightarrow y' = \frac{\frac{1}{2}(2-x^4)^{-1/2}(-4x^3)\,x - \sqrt{2-x^4}}{x^2} = \frac{\frac{-2x^4}{\sqrt{2-x^4}} - \sqrt{2-x^4}}{x^2}$$

$$y' = \frac{-x^4-2}{x^2\sqrt{2-x^4}} \,.\quad \text{So } (2x^2+y^2) + xyy' = 2x^2 + \frac{2-x^4}{x^2} - x\,\frac{\sqrt{2-x^4}}{x}\,\frac{(x^4+2)}{x^2\sqrt{2-x^4}}$$

$$= 2x^2 + \frac{2-x^4}{x^2} - \frac{x^4+2}{x^2} = \frac{2x^4 + 2 - x^4 - x^4 - 2}{x^2} = 0$$

3) $$\underbrace{(y^3 + 3x^2\cos y)}_{M}\,dx + \underbrace{(3xy^2 - x^3\sin y)}_{N}\,dy = 0 \;;\quad y(1) = 0$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 3y^2 - 3x^2\sin y\\ \frac{\partial N}{\partial x} &= 3y^2 - 3x^2\sin y\end{aligned}\right\}$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$: the ODE is exact. We look for a function $F(x,y)$ such that $\dfrac{\partial F}{\partial x} = M$, $\dfrac{\partial F}{\partial y} = N$

$$\frac{\partial F}{\partial x} = y^3 + 3x^2\cos y \Rightarrow F(x,y) = xy^3 + x^3\cos y + h(y) \Rightarrow$$

$\dfrac{\partial F}{\partial y} = 3xy^2 - x^3\sin y + h'(y)$. But $\dfrac{\partial F}{\partial y} = N = 3xy^2 - x^3\sin y \Rightarrow h'(y) = 0$

$\Rightarrow h(y) = K$, a constant.

Additional Page

The general solution is $F(x,y) = \text{Constant} \Rightarrow \quad xy^3 + x^3 \cos y = C$.

$y(1) = 0 \Rightarrow C = 1$. The particular solution is then $xy^3 + x^3 \cos y = 1$.

<u>Verification (Not required)</u> We take implicit differentiation of $xy^3 + x^3 \cos y = 1$ with respect to x:

$$y^3 + 3xy^2 y' + 3x^2 \cos y - x^3 \sin y \cdot y' = 0 \Rightarrow$$

$(y^3 + 3x^2 \cos y)\,dx + (3xy^2 - x^3 \sin y)\,dy = 0$: this is the original ODE

Question 2. [12 points] Solve each of the following IVPs.

1. $(2+3xy+y^2)dx + (x^2+xy)dy = 0, \quad y(1) = -1.$
2. $\left(\ln(y^4) - y^2 + 4x\ln y\right)dx + \left(\frac{4x}{y} - 2y\right)dy = 0, \quad y(0) = 1.$
3. $(y + 2xy^3 + 2xy^4e^y)dx + (-3x + x^2y^4e^y - x^2y^2)dy = 0; \quad y(0) = 1$

1) $\frac{\partial M}{\partial y} = 3x + 2y$; $\frac{\partial N}{\partial x} = 2x + y$: Not exact since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x+y}{x(x+y)} = \frac{1}{x}$$; an integrating factor exists and

given by $\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$. We multiply the ODE by x:

$$\underbrace{(2x + 3x^2y + xy^2)}_{M^*}dx + \underbrace{(x^3 + x^2y)}_{N^*}dy = 0 \;;$$

$\left.\begin{array}{l}\frac{\partial M^*}{\partial y} = 3x^2 + 2xy \\ \frac{\partial N^*}{\partial x} = 3x^2 + 2xy\end{array}\right\}$ $\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$ and the ODE is exact. We look for a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M^*$, $\frac{\partial F}{\partial y} = N^*$

$$\frac{\partial F}{\partial y} = x^3 + x^2y \Rightarrow F(x,y) = x^3y + \frac{1}{2}x^2y^2 + h(x) \Rightarrow \frac{\partial F}{\partial x} = 3x^2y + xy^2 + h'(x)$$

$$\frac{\partial F}{\partial x} = M^* \Rightarrow 3x^2y + xy^2 + h'(x) = 2x + 3x^2y + xy^2 \Rightarrow h'(x) = 2x \Rightarrow$$

$h(x) = x^2 + K$, so $F(x,y) = x^3y + \frac{1}{2}x^2y^2 + x^2 + K$. The general solution is then $x^3y + \frac{1}{2}x^2y^2 + x^2 = C$. Now $y(1) = -1 \Rightarrow -1 + \frac{1}{2} + 1 = C \Rightarrow C = 1/2$

The particular solution is then $\boxed{x^3y + \frac{1}{2}x^2y^2 + x^2 = \frac{1}{2}}$

Verification (Not required) We take implicit differentiation:

$$3x^2y + x^3y' + xy^2 + x^2yy' + 2x = 0 \Rightarrow (2x + 3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$$

2) $\dfrac{\partial M}{\partial y} = \dfrac{4}{y} - 2y + \dfrac{4x}{y}$; $\dfrac{\partial N}{\partial x} = \dfrac{4}{y}$: Not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2y + \frac{4x}{y}}{\frac{4x}{y} - 2y} = 1$$; an integrating factor exists and given

by $\mu(x) = e^{\int 1\,dx} = e^{x}$. Multiply the ODE by e^{x}:

$$\underbrace{\left(e^{x}\ln(y^{4}) - e^{x}y^{2} + 4xe^{x}\ln y\right)}_{M^*} dx + \underbrace{\left(\frac{4xe^{x}}{y} - 2ye^{x}\right)}_{N^*} dy = 0$$

$\dfrac{\partial M^*}{\partial y} = \dfrac{4e^{x}}{y} - 2e^{x}y + \dfrac{4xe^{x}}{y}$

$\dfrac{\partial N^*}{\partial x} = \dfrac{4e^{x}}{y} + \dfrac{4xe^{x}}{y} - 2ye^{x}$

$\dfrac{\partial M^*}{\partial y} = \dfrac{\partial N^*}{\partial x}$: the ODE is now Exact! We look for a function $F(x,y)$ such that $\dfrac{\partial F}{\partial x} = M^*$ and $\dfrac{\partial F}{\partial y} = N^*$

$\dfrac{\partial F}{\partial y} = \dfrac{4xe^{x}}{y} - 2ye^{x} \Rightarrow F(x,y) = 4xe^{x}\ln(y) - y^{2}e^{x} + h(x) \Rightarrow$

$\dfrac{\partial F}{\partial x} = 4e^{x}\ln y + 4xe^{x}\ln y - y^{2}e^{x} + h'(x)$. But $\dfrac{\partial F}{\partial x} = M^* \Rightarrow$

$$4e^{x}\ln y + 4xe^{x}\ln y - y^{2}e^{x} + h'(x) = \underbrace{e^{x}\ln(y^{4})}_{=4e^{x}\ln y} - e^{x}y^{2} + 4xe^{x}\ln y \Rightarrow h'(x) = 0$$

$\Rightarrow h(x) = k =$ Constant. So $F(x,y) = 4xe^{x}\ln(y) - y^{2}e^{x} + k$ and the general solution is $4xe^{x}\ln(y) - y^{2}e^{x} = C$. Now $y(0) = 1 \Rightarrow C = -1$

The particular solution is $\boxed{4xe^{x}\ln(y) - y^{2}e^{x} = -1}$

<u>Verification (Not required)</u> Take implicit differentiation:

$$4e^{x}\ln(y) + 4xe^{x}\ln y + \frac{4xe^{x}}{y}y' - 2yy'e^{x} - y^{2}e^{x} = 0 \Rightarrow$$

$$\left(\underbrace{4e^{x}\ln(y)}_{=e^{x}\ln(y^{4})} + 4xe^{x}\ln(y) - y^{2}e^{x}\right)dx + \left(\frac{4x}{y}e^{x} - 2ye^{x}\right)dy = 0$$; this is the ODE $M^*\,dx + N^*\,dy = 0$

3) $\frac{\partial M}{\partial y} = 1 + 6xy^2 + 8xy^3 e^y + 2xy^4 e^y$; $\frac{\partial N}{\partial x} = -3 + 2xy^4 e^y - 2xy^2$: Not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{4 + 8xy^2 + 8xy^3 e^y}{y(1 + 2xy^2 + 2xy^3 e^y)} = \frac{4(1 + 2xy^2 + 2xy^3 e^y)}{y(1 + 2xy^2 + 2xy^3 e^y)} = \frac{4}{y} = g(y) \mid \text{on}$$

integrating factor exists and given by $\mu(y) = e^{-\int g(y)\,dy} = e^{-4\ln y} = \frac{1}{y^4}$.

Multiply the ODE by $\frac{1}{y^4}$:

$$\underbrace{\left(\frac{1}{y^3} + \frac{2x}{y} + 2xe^y\right)}_{M^*} dx + \underbrace{\left(-\frac{3x}{y^4} + x^2e^y - \frac{x^2}{y^2}\right)}_{N^*} dy = 0$$

$\frac{\partial M^*}{\partial y} = -\frac{3}{y^4} - \frac{2x}{y^2} + 2xe^y$; $\frac{\partial N^*}{\partial x} = -\frac{3}{y^4} + 2xe^y - \frac{2x}{y^2}$; Exact.

Look for a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M^*$ and $\frac{\partial F}{\partial y} = N^*$

$\frac{\partial F}{\partial x} = \frac{1}{y^3} + \frac{2x}{y} + 2xe^y \Rightarrow F(x,y) = \frac{x}{y^3} + \frac{x^2}{y} + x^2e^y + h(y) \Rightarrow$

$\frac{\partial F}{\partial y} = -\frac{3x}{y^4} - \frac{x^2}{y^2} + x^2e^y + h'(y)$. But $\frac{\partial F}{\partial y} = N^* = -\frac{3x}{y^4} + x^2e^y - \frac{x^2}{y^2} \Rightarrow$

$h'(y) = 0 \Rightarrow h(y) = K = $ Constant. So $F(x,y) = \frac{x}{y^3} + \frac{x^2}{y} + x^2e^y + K$. The general solution is $\frac{x}{y^3} + \frac{x^2}{y} + x^2e^y = C$

$y(1) = 1 \Rightarrow 2 + e = C$. The particular solution is $\frac{x}{y^3} + \frac{x^2}{y} + x^2e^y = 2 + e$

<u>Verification (Not required)</u> Take implicit differentiation:

$$\frac{y^3 - 3y^2y'x}{y^6} + \frac{2xy - x^2y'}{y^2} + 2xe^y + x^2e^y y' = 0 \Rightarrow$$

$$y^3 - 3y^2y'x + 2xy^5 - x^2y^4y' + 2xy^6e^y + x^2e^yy^6y' = 0 \Rightarrow$$

$$(y^3 + 2xy^5 + 2xy^6e^y)\,dx + (-3xy^2 - x^2y^4 + x^2e^yy^6)\,dy = 0 \Rightarrow$$

$$(y + 2xy^3 + 2xy^4e^y)\,dx + (-3x - x^2y^2 + x^2e^yy^4)\,dy = 0 \quad \text{(after dividing with } y^2\text{)}$$

Question 3. [5 points] Consider the function $f(x) = x^4 + 6x - 5$.

(a) (**[1 point]**) Use the intermediate Value Theorem to prove that $f(x)$ has a root in the interval $[0, 1]$.

(b) (**[2 points]**) Rewrite the equation $f(x) = 0$ under the form $g(x) = x$ and verify that the function $g(x)$ verify the two conditions for the convergence of the iteration sequence $x_{n+1} = g(x_n)$.

(c) (**[2 points]**) Use the **fixed point iteration** method to find the root of the function $f(x)$ in the interval $[0, 1]$ to 5 decimal places, starting with $x_0 = 0.75$.

(a) Clearly, $f(x)$ is Continuous on $[0,1]$. Moreover, $f(0)=-5<0$ and $f(1)=2>0$. By the intermediate Value Theorem, f must have a root in $[0,1]$

(b) $x^4+6x-5=0 \Rightarrow x = \frac{5-x^4}{6}$. Let $g(x)=\frac{5-x^4}{6}$, then clearly $g(x)$ is Continuous on $[0,1]$. Moreover, $g'(x) = -\frac{2x^3}{3}$ is also continuous and $|g'(x)| = \frac{2}{3}|x^3| \leq \frac{2}{3} < 1$ on $[0,1]$. By a Theorem seen in class, the iteration sequence $x_{n+1} = g(x_n)$ converges for any choice of x_0.

$x_0 = 0.75 \Rightarrow x_1 = g(x_0) = \frac{5-0.75^4}{6} = 0.78060$

$x_2 = g(x_1) = \frac{5-0.78060^4}{6} = 0.77145$; $x_3 = g(x_2) = \frac{5-0.77145^4}{6} = 0.77430$

$x_4 = g(x_3) = \frac{5-0.77430^4}{6} = 0.77343$; $x_5 = g(x_4) = \frac{5-0.77343^4}{6} = 0.77369$

$x_6 = g(x_5) = \frac{5-0.77369^4}{6} = 0.77361$; $x_7 = g(x_6) = \frac{5-0.77361^4}{6} = 0.77361$

So the root of the equation $x^4+6x-5=0$ is $x \approx 0.77361$ correct to 5 decimal places!

Question 4. [4 points]

(a) ([**2 point**]) Use Newton's Method to estimate the value of $\sqrt[4]{5}$ to 5 decimal places. Start with $x_0 = 1.25$.

(b) ([**2 point**]) Use Newton's Method to find the first coordinate of the intersection point of the two curves $y = x^3$ and $y = \cos(x)$ to 5 decimal places. Start with $x_0 = 1$.

(a) let $x = \sqrt[4]{5}$, then $x^4 = 5 \Rightarrow x^4 - 5 = 0$. The function $f(x) = x^4 - 5$ is clearly continuous. The same is true for $f'(x) = 4x^3$

$$x_0 = 1.25 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.25 - \frac{(1.25)^4 - 5}{4(1.25)^3} = 1.5775$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5775 - \frac{(1.5775)^4 - 5}{4(1.5775)^3} = 1.50155$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.50155 - \frac{(1.50155)^4 - 5}{4(1.50155)^3} = 1.49539$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.49539 - \frac{(1.49539)^4 - 5}{4(1.49539)^3} = 1.49535$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.49535 - \frac{(1.49535)^4 - 5}{4(1.49535)^3} = 1.49535$$

So $\sqrt[4]{5} = 1.49535$ rounded to 5 decimal places

(b) $\cos x = x^3 \Leftrightarrow \cos x - x^3 = 0$. let $g(x) = \cos x - x^3$; clearly g is a continuous function and $g'(x) = -\sin x - 3x^2$ is also continuous!

$$x_0 = 1 \Rightarrow x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = 1 - \frac{\cos(1) - 1^3}{-\sin(1) - 3(1)^2} = 0.88033$$

$$x_2 = x_1 - \frac{g(x_1)}{g'(x_1)} = 0.88033 - \frac{\cos(0.88033) - (0.88033)^3}{-\sin(0.88033) - 3(0.88033)^2} = 0.86568$$

$$x_3 = x_2 - \frac{g(x_2)}{g'(x_2)} = 0.86568 - \frac{\cos(0.86568) - (0.86568)^3}{-\sin(0.86568) - 3(0.86568)^2} = 0.86547$$

$$x_4 = x_3 - \frac{g(x_3)}{g'(x_3)} = 0.86547 - \frac{\cos(0.86547) - (0.86547)^3}{-\sin(0.86547) - 3(0.86568)^2} = 0.86547$$

We conclude that the first coordinate of the intersection point

is $x \simeq 0.86547$