MAT2384-Ordinary differential Equations and Numerical Methods-Fall 2010

MIDTERM EXAM (October 22, 2010)

Professor: Joseph Khoury **Duration: 80 minutes**

Last Name: Solution

First Name: ____________

Student Number: ____________

(1) This is a closed book exam.

(2) Only basic scientific calculators are allowed. Graphing or programmable calculators are not permitted.

(3) The exam has 6 questions worth a total of 30 points.

(4) The exam has 9 pages.

(5) Please write your answers in a complete and clear way. You may use the back of the pages or the additional pages at the end if you need more space for your work.

(6) You must answer all the questions.

MAT 2384 C **Midterm Exam, October 22, 2010**

1. **[7 points]** Solve the Initial Value Problem:

$$\underbrace{(6xy^2 - 3x^2y^3 + 2y)}_{M}dx + \underbrace{(9x^2y - 4x^3y^2 + 4x)}_{N}dy = 0, \quad y(1) = 1.$$

Solution $M_y = 12xy - 9x^2y^2 + 2$, $N_x = 18xy - 12x^2y^2 + 4$: NOT Exact

$$\frac{M_y - N_x}{M} = \frac{-6xy + 3x^2y^2 - 2}{-y(-6xy + 3x^2y^2 - 2)} = -\frac{1}{y} = g(y).$$

The integrating factor is $\mu(y) = e^{-\int(-\frac{1}{y})dy} = e^{\ln y} = y$.

Multiply the ODE with y:

$$\underbrace{(6xy^3 - 3x^2y^4 + 2y^2)}_{M^*}dx + \underbrace{(9x^2y^2 - 4x^3y^3 + 4xy)}_{N^*}dy = 0$$

$M^*_y = 18xy^2 - 12x^2y^3 + 4y$, $N^*_x = 18xy^2 - 12x^2y^3 + 4y$; Now Exact

Look for a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M^*$, $\frac{\partial F}{\partial y} = N^*$

$\frac{\partial F}{\partial x} = M^* = 6xy^3 - 3x^2y^4 + 2y^2 \Rightarrow F(x,y) = 3x^2y^3 - x^3y^4 + 2xy^2 + h(y)$

$\Rightarrow \frac{\partial F}{\partial y} = 9x^2y^2 - 4x^3y^3 + 4xy + h'(y) = N^* \Rightarrow h'(y) = 0$

$\Rightarrow h(y) = K$. So $F(x,y) = 3x^2y^3 - x^3y^4 + 2xy^2 + K$.

The general solution is $3x^2y^3 - x^3y^4 + 2xy^2 = C$

Now $y(1) = 1 \Rightarrow 3 - 1 + 2 = C \Rightarrow C = 4$.

The unique solution is $\boxed{3x^2y^3 - x^3y^4 + 2xy^2 = 4}$

2. **[4 points]** Solve the Initial Value Problem:

$$y' - \frac{2}{x}y = 2x^3, \qquad y(1) = 0.$$

<u>Solution</u> This is a first-order linear ODE with

$f(x) = -\frac{2}{x}$, $r(x) = 2x^3$. The general solution is

$$y(x) = \frac{\int e^{\int -\frac{2}{x}dx} \cdot 2x^3 dx + C}{e^{\int -\frac{2}{x}dx}} = \frac{\int e^{-2\ln x} 2x^3 dx + C}{e^{-2\ln x}}$$

$$= \frac{\int \frac{1}{x^2} \cdot 2x^3 dx + C}{1/x^2} = x^2\left[2\int x dx + C\right] =$$

$$x^2(x^2 + C) = x^4 + Cx^2$$

Now $y(1) = 0 \Rightarrow 1 + C = 0 \Rightarrow C = -1$

So the unique solution is $\boxed{y(x) = x^4 - x^2}$

3. [**4 points**] Give the general solution for each of the following ODEs:

(1) $y'' - 6y' + 9y = 0$

(2) $y'' - 4y' + 5y = 0$

Solution (1) The characteristic equation is

$\lambda^2 - 6\lambda + 9 = 0 \iff (\lambda - 3)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 3$ a double real root. The general solution is $\boxed{y(x) = C_1 e^{3x} + C_2 x e^{3x}}$

(2) The characteristic equation is $\lambda^2 - 4\lambda + 5 = 0 \Rightarrow$

$\lambda_1 = \frac{4 + \sqrt{16 - 4(1)(5)}}{2} = \frac{4 + \sqrt{-4}}{2} = \frac{4 + 2i}{2} = 2 + i$ and

$\lambda_2 = 2 - i$: 2 complex conjugate roots.

The general solution is

$$\boxed{y(x) = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x}$$

4. [**5 points**] Solve the Initial Value Problem:

$$x^2y'' + xy' + 9y = 0, \quad x > 0, \ y(1) = 1, \ y'(1) = 3$$

<u>Solution</u> This is an Euler – Cauchy Equation with $a=1$, $b=9$. The characteristic equation is

$m^2 + (a-1)m + b = 0 \iff m^2 + 9 = 0 \Rightarrow m_1 = 3i, \ m_2 = -3i$

2 complex conjugate roots. The general solution is

$$\boxed{y(x) = C_1 \cos(3\ln x) + C_2 \sin(3\ln x)}$$

Now $y'(x) = -\frac{3}{x} C_1 \sin(3\ln x) + \frac{3}{x} C_2 \cos(3\ln x)$

$y(1) = 1 \Rightarrow C_1 = 1$

$y'(1) = 3 \Rightarrow 3C_2 = 3 \Rightarrow C_2 = 1$

The unique solution is $\boxed{y(x) = \cos(3\ln x) + \sin(3\ln x)}$

5. [**4 points**] Give the general solution for the following ODE:

$$y''' + 3y'' - y' - 3y = 0.$$

Solution The characteristic equation is

$\lambda^3 + 3\lambda^2 - \lambda - 3 = 0 \Rightarrow \lambda^2(\lambda+3) - (\lambda+3) = 0 \Leftrightarrow$

$(\lambda+3)(\lambda^2-1) = 0 \Rightarrow \lambda_1 = -3,\ \lambda_2 = -1,\ \lambda_3 = 1$

3 distinct real roots.

The general solution is $\boxed{y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 e^{x}}$

6. [**6 points**] Use the **fixed point iteration** method to find the root of

$$x^4 - 7x + 3 = 0$$

in the interval $[0,1]$ to 4 decimal places. Use $x_0 = 0.45$ and **make sure that the conditions for convergence of the iteration sequence are satisfied.**

Solution let $f(x) = x^4 - 7x + 3$, then f is clearly continuous and $f(0) = 3 > 0$, $f(1) = -3 < 0$. By the intermediate value Theorem, f must have a root in $[0,1]$.

$x^4 - 7x + 3 = 0 \iff x = \frac{x^4+3}{7}$. Let $g(x) = \frac{x^4+3}{7}$. Clearly, g is continuous and $g'(x) = \frac{4x^3}{7}$

$|g'(x)| = \frac{4}{7}|x^3| \leq \frac{4}{7}$ on $[0,1]$. As $\frac{4}{7} < 1$, we know that the Iteration sequence will convergence.

$$x_0 = 0.45 \Rightarrow x_1 = g(x_0) = \frac{(0.45)^4+3}{7} = 0.4344$$

$$x_2 = g(x_1) = \frac{(0.4344)^4+3}{7} = 0.4337$$

$$x_3 = g(x_2) = \frac{(0.4337)^4+3}{7} = 0.4336$$

$$x_4 = g(x_3) = \frac{(0.4336)^4+3}{7} = 0.4336 \quad \text{stop.}$$

So $x \simeq 0.4336$ to 4 decimal places