# CHAPTER 5. INTEGRALS

## § 5.1, 5.2. AREAS, DISTANCES, AND DEFINITE INTEGRALS

### *1. Introducing the Definite Integral as the Area of a Region*

Let $y = f(x) \geq 0$ be a function defined in a closed interval $[a, b]$. Let $R$ be the region under the graph of $f(x)$ and above the $x$-axis in the interval $[a, b]$.

Let $h = \dfrac{b-a}{n}$, and let $x_i = a + ih$, $i = 0, 1, 2, \ldots, n$. Then $x_0 = a, x_1, x_2, \ldots, x_n = b$, called *mesh points*, subdivide the interval $[a, b]$ into $n$ subintervals each with length $h$. Taking an arbitrary value $x_i^*$ in each interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, the sum

$S(n) = f(x_1^*)h + f(x_2^*)h + \ldots + f(x_n^*)h = h(f(x_1^*) + f(x_2^*) + \ldots + f(x_n^*))$,

called a *Riemann Sum*, can be reasonably regarded as an approximation of the "area" of the region $R$.

When $n$ approaches infinity, the limit of the Riemann sums, if it exists, is **defined to be** the *area* of the region $R$. This limit is called the *definite integral* of $f(x)$ on the interval $[a, b]$ and denoted by $\int_a^b f(x)dx$, where $a$ is the *lower limit*, $b$ is the *upper limit*, $f(x)$ is the *integrand*, and $x$ is the *variable of integration*.

Since a definite integral is a number, the variable of integration can be replaced by any other symbol. The variable of integration of a definite integral is a *dummy variable*.

If $f(x) \leq 0$ in $[a, b]$, we can still define the definite integral of function $f(x)$ on interval $[a, b]$ in the same way, but the area of the region **under** the $x$-axis and **above** the graph of $f(x)$ is defined to be the negation of the definite integral. If some part of the graph of $f(x)$ is above the $x$-axis, and some part of the graph of $f(x)$ is under the $x$-axis in $[a, b]$, we have to calculate the area of each part separately and then add them up. The sum is the area between the graph of $f(x)$ and the $x$-axis on this interval.

Since the definite integral calculates the area of a region, if we know the area of the region, we can find the definite integral.

*Examples*

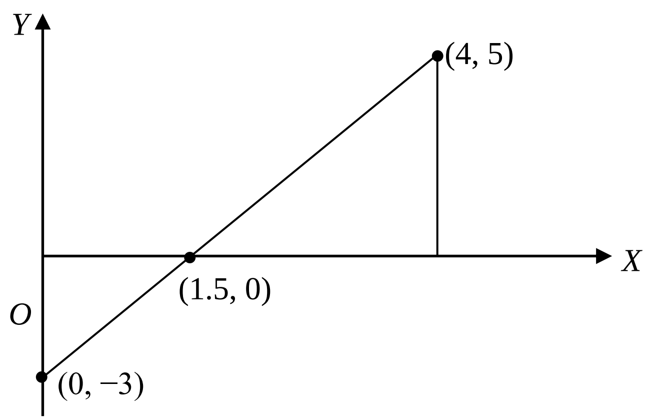
5.1.1. Find the definite integral $\int_{-R}^{R} \sqrt{R^2 - x^2}\,dx$, $R > 0$, by the area of a region.

The graph of the function $y = \sqrt{R^2 - x^2}$ is the upper half of a circle centered at the origin with radius $R$. This definite integral is the area of the half-disk under the upper half of this circle and above the $x$-axis. The area of this half-disk is $A = \pi R^2 / 2$. Hence

$$\int_{-R}^{R} \sqrt{R^2 - x^2}\,dx = \frac{\pi R^2}{2}.$$

5.1.2. Find $\int_0^4 (2x - 3)dx$.

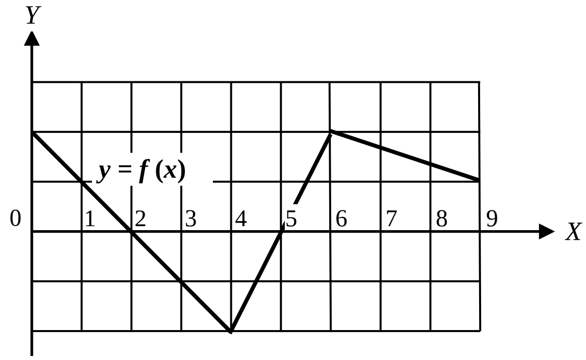
The graph of the function $y = 2x - 3$, $0 \le x \le 4$, is a line segment joining point $(0, -3)$ and point $(4, 5)$. This line segment intersects the $x$-axis at point $(1.5, 0)$.




When $0 \le x < 1.5$, $2x - 3 < 0$; when $1.5 < x \le 4$, $2x - 3 > 0$. The area between the graph of $2x - 3$ and the $x$-axis in interval $[0, 1.5]$ is $\frac{1}{2} \times 3 \times \frac{3}{2} = \frac{9}{4} = 2.25$, and the area between the graph of $2x - 3$ and the $x$-axis in interval $[1.5, 4]$ is $\frac{1}{2} \times 5 \times \frac{5}{2} = \frac{25}{4} = 6.25$. Note that the integral is the negation of the area of a region under the $x$-axis, i.e, $\int_{1.5}^{4} (2x - 3)dx = -2.25$. Hence,

$$\int_0^4 (2x - 3)dx = \int_0^{1.5} (2x - 3)dx + \int_{1.5}^{4} (2x - 3)dx = 6.25 - 2.25 = 4.$$

5.1.3. Suppose the graph of a function $y = f(x)$, $0 \le x \le 9$, is given in the following figure:




By calculating the area of the regions between the graph of this function and the *x*-axis, we have

$\int_0^2 f(x)dx = 2$, $\int_2^5 f(x)dx = -3$, and $\int_5^9 f(x)dx =$ 5.5. Hence $\int_0^9 f(x)dx =$ 4.5.

The following example uses the definition of the definite integral to find the area of a curved region:

5.1.4. Find the area of the region under the graph of the function $y = x^2$ and above the *x*-axis in interval [0, 1].

This area is given by the definite integral $\int_0^1 x^2 dx$ .

Subdivide the interval [0, 1] into $n$ subintervals with equally spaced mesh points $x_0 = 0$, $x_1 = \frac{1}{n}$, $x_2 = \frac{2}{n}$, …, $x_n = 1$. Then $h = \frac{1}{n}$. Let $x_i^* = x_i = \frac{i}{n}$, $i = 1, 2, \ldots, n$.

$$\int_0^1 x^2 dx = \lim_{n\to\infty}\frac{1}{n}\left(\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+\left(\frac{n}{n}\right)^2\right)=\lim_{n\to\infty}\frac{1}{n^3}\left(1^2+2^2+\ldots n^2\right)=\lim_{n\to\infty}\frac{1}{n^3}\left(\frac{1}{6}n(n+1)(2n+1)\right)$$

$$=\lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3}=\frac{1}{3}.$$

***2. Properties of the Definite Integral***

(i) $\int_a^a f(x)dx = 0$.

(ii) $\int_a^b c\,dx = c(b-a)$.

(iii) $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

(iv) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

(v) $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

(vi) $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$.

(vii) If $f(x) \geq g(x)$ for all $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

This property has the following immediate corollaries:

(vii-a) If $f(x) \geq 0$ for all $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.

(vii-b) If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

*Example* 5.1.5. Find $\int_4^{10} (3f(x)-2)dx$ if $\int_1^4 f(x)dx = -2$ and $\int_1^{10} f(x)dx = 3$.

$\int_4^{10} (3f(x)-2)dx = 3\left(\int_1^{10} f(x)dx - \int_1^4 f(x)dx\right) - 2\int_4^{10} dx = 3(3 + 2) - 12 = 3$.

*Example* 5.1.6. If $\int_1^5 f(x)dx = 5$, $\int_3^{10} f(x)dx = 7$, and $\int_1^{10} f(x)dx = 10$, find $\int_3^5 f(x)dx$.

Since $\int_1^5 f(x)dx = \int_1^3 f(x)dx + \int_3^5 f(x)dx$, $\int_3^5 f(x)dx = \int_1^5 f(x)dx - \int_1^3 f(x)dx$.

Since $\int_1^3 f(x)dx + \int_3^{10} f(x)dx = \int_1^{10} f(x)dx$, $\int_1^3 f(x)dx = \int_1^{10} f(x)dx - \int_3^{10} f(x)dx$.

Hence, $\int_3^5 f(x)dx = \int_1^5 f(x)dx - \int_1^{10} f(x)dx + \int_3^{10} f(x)dx = 5 - 10 + 7 = 2$

If the function $v(t)$ is the velocity function of a particle moving along the $x$-axis, then the definite integral $\int_a^b v(t)dt$ is the displacement of the particle from $t = a$ to $t = b$.

## § 5.3, 5.4. FUNDAMENTAL THEOREM OF CALCULUS

### *1. Definite Integral with a Variable Upper Limit*

When the upper and lower limits are given, the definite integral $\int_a^b f(t)dt$ is a number. This number changes when the upper limit changes. In other words, a definite integral with a variable upper limit is a function of the upper limit: $I(x) = \int_a^x f(t)dt$.

*Example* 5.2.1. Consider function $f(x) = 2x$. The area of the triangle between the graph of this function and the $x$-axis in interval $[0, a]$ is $a^2$. Let $a$ be a variable. We have a function

$I(a) = \int_0^a (2x)dx = a^2$.

### *2. Differentiating a Function Defined by a Definite Integral with Variable Limits*

Consider function $I(x) = \int_a^x f(t)dt$, where $f(x)$ is continuous on an interval $[a, b]$, $a \le x \le b$.
Then $I(x+h) - I(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$.

Let $M(x, h) = \max\{f(t);\ t \text{ is between } x \text{ and } x+h\}$ and let $m(x, h) = \min\{f(t);\ t \text{ is between } x \text{ and } x+h\}$. Then

$M(x, h)h \le I(x+h) - I(x) \le m(x, h)h$, and $M(x, h) \le \frac{1}{h}(I(x+h) - I(x)) \le m(x, h)$. When $h$ approaches zero, $\lim_{h\to 0} M(x, h) = \lim_{h\to 0} m(x, h) = f(x)$. Hence, $\lim_{h\to 0} \frac{1}{h}(I(x+h) - I(x)) = f(x)$, or
$\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

*Fundamental Theorem of Calculus* (*Part I*). If $f(x)$ is continuous on an interval $[a, b]$ and $a \le x \le b$, the derivative of the function $I(x) = \int_a^x f(t)dt$ is $f(x)$.

By the properties of the definite integral and the chain rule, we also have

$$\frac{d}{dx}\int_x^a f(t)dt = -\frac{d}{dx}\int_a^x f(t)dt = -f(x),$$

$$\frac{d}{dx}\int_a^{h(x)} f(t)dt = \left(\frac{d}{du}\int_a^u f(t)dt\right)\left(\frac{du}{dx}\right) = f(u)u_t' = f(h(x))h'(x), \text{ where } u = h(x).$$

$\frac{d}{dx}\int_{h(x)}^{a} f(t)dt = -\frac{d}{dx}\int_{a}^{h(x)} f(t)dt = -f(u)u_t' = -f(h(x))h'(x)$, where $u = h(x)$,

$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = \frac{d}{dx}\left(\int_{a}^{h(x)} f(t)dt + \int_{g(x)}^{a} f(t)dt\right) = f(h(x))\, h'(x) - f(g(x))g'(x).$

*Examples*

5.2.2. $\frac{d}{dx}\int_0^x e^{2t^2}dt = e^{2x^2}$ .

5.2.3. $\frac{d}{du}\int_u^{-1} e^{x^2}dx = -e^{u^2}$ .

5.2.4. $\frac{d}{dx}\int_0^{e^x} \cos(t^2)dt = \left(\frac{d}{du}\int_a^u \cos(t^2)dt\right)\left(\frac{de^x}{dx}\right) = e^x\cos(u^2) = e^x\cos(e^{2x})$. .

5.2.5. $\frac{d}{dx}\int_{\arctan x}^{0} \sin(t^2)dt = -\frac{d}{dx}\int_0^{\arctan x}\sin(t^2)dt = -\frac{\sin((\arctan x)^2)}{1+x^2}$ .

5.2.6. $\int_{x^3}^{x^2} \sin(t^2)dt = 2x\sin(x^4) - 3x^2\sin(x^6)$ .

### *3. Finding Definite Integrals by Antiderivatives*

Recall that the difference between two antiderivatives of a function is a constant. If $F(x)$ is an antiderivative of $f(x)$, then function $I(x) = \int_a^x f(t)dt = F(x) + C$ . Let $x = a$. We have $C = -F(a)$. Hence,

$$\int_a^x f(t)dt = F(x) - F(a) = \left[F(t)\right]_{t=a}^{x} .$$

Let $x = b$, and use $x$ as the dummy variable. We have the second part of the Fundamental Theorem of Calculus:

*Fundamental Theorem of Calculus* (*Part II*). If $f(x)$ is continuous on an interval $[a, b]$, then $\int_a^b f(x)dx = F(a) - F(b)$ , where $F(x)$ is an antiderivative of $f(x)$.

Since $f(x) = F'(x)$, this formula can also be written as $\int_a^b F'(x)dx = F(b) - F(a)$, called the *Net Change Theorem*. I.e., the *net change* of function $F(x)$ from $x = a$ to $x = b$, $F(b) - F(a)$, is the definite integral of its derivative $F'(x)$ from $x = a$ to $x = b$.

*Examples*

5.2.7. From section 4.9, we see that $F(x) = \arctan x$ is an antiderivative of the function $f(x) = \dfrac{1}{1+x^2}$. Hence

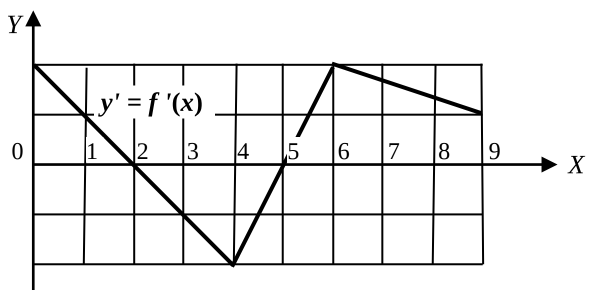
$$\int_0^1 \frac{1}{1+x^2}dx = [\arctan x]_{x=0}^1 = \frac{\pi}{4}.$$

5.2.8. Suppose the derivative of function $F(x)$ is $F'(x) = 3x^2$ such that $F(1) = 3$, what is $F(3)$?

$$F(3) - F(1) = \int_1^3 3x^2 dx = \left[x^3\right]_{x=1}^3 = 26.$$

Hence, $F(3) = 26 + F(1) = 26 + 3 = 29$.

5.2.9. Suppose the derivative of a function $f(x)$ is given in the following figure. Sketch the graph of this function if $f(0) = 0$.




By counting the area under the graph of the derivative $f'(x)$, we see that

$f(2) = f(0) + \int_0^2 f'(x)dx = 0 + 2 = 2,$

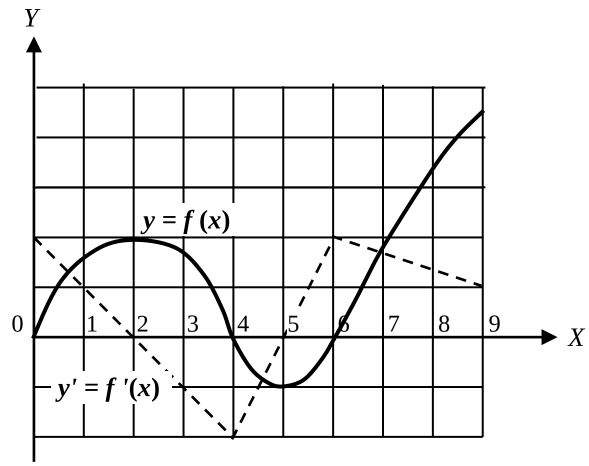
$f(4) = f(2) + \int_2^4 f'(x)dx = 2 + (-2) = 0,$

$f(5) = f(4) + \int_4^5 f'(x)dx = 0 + (-1) = -1,$

$f(6) = f(5) + \int_5^6 f'(x)dx = -1 + 1 = 0,$

$f(9) = f(6) + \int_6^9 f'(x)dx = 0 + 4.5 = 4.5.$

In interval (0, 2), $f'(x) > 0$ and decreasing, $f(x)$ is increasing and concave down; in interval (2, 4), $f'(x) > 0$ and decreasing, $f(x)$ is decreasing and concave down; in interval (4, 5), $f'(x) < 0$ and increasing, $f(x)$ is decreasing and concave up; in interval (5, 6), $f'(x) > 0$ and increasing, $f(x)$ is increasing and concave up; in interval (6, 9), $f'(x) > 0$ and decreasing, $f(x)$ is increasing and concave down. Hence, the graph of the function $f(x)$ looks like the following:




## § 5.5. THE SUBSTITUTION RULE

### *1. The Method of Variable Substitution*

Suppose we want to find an indefinite integral $\int f(x)dx$.

Step 1. Define an *intermediate variable* $u = g(x)$.

Step 2. Multiply the integrand by the reciprocal of $g'(x)$ and change $dx$ to $du$. In other words, replace $dx$ by $\frac{1}{g'(x)}du$.

Step 3. Convert $\frac{f(x)}{g'(x)}$ as a function of $u$: $\frac{f(x)}{g'(x)} = h(u)$.

Step 4. Find $\int h(u)du = H(u) + C$.

Step 5. $\int f(x)dx = H(g(x)) + C$.

This procedure can be summarized as

$\int f(x)dx = \int f(x)\dfrac{1}{g'(x)}du = \int h(u)du = H(u) + C = H(g(x)) + C$, where $u = g(x)$,

$h(u) = \dfrac{f(x)}{g'(x)}$, and $H(u)$ is an antiderivative of $h(u)$.

From these steps we can see two rules of thumb to define the intermediate variable, which are

(a) $\dfrac{f(x)}{g'(x)}$ should be easy to convert to a function $h(u)$ of $u$, and

(b) the integral $\int h(x)du$ should be easy to integrate.

Otherwise, you may have chosen a wrong function $g(x)$ or this method cannot be used to solve this question.

*Examples*

5.3.1. $\int x\sqrt{x^2-1}dx$

Let $u = g(x) = x^2 - 1$. $g'(x) = 2x$.

$$\int x\sqrt{x^2-1}dx = \int \frac{x\sqrt{x^2-1}}{2x}du = \int \frac{\sqrt{x^2-1}}{2}du = \int \frac{\sqrt{u}}{2}du = \frac{1}{2}\int u^{1/2}du = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2} + C$$
$$= \frac{1}{3}(x^2-1)^{3/2} + C.$$

5.3.2. $\int e^{-2x}dx$.

Let $u = -2x$, $u' = -2$.

$$\int e^{-2x}dx = \int e^{-2x}\left(-\frac{1}{2}\right)du = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C == -\frac{1}{2}e^{-2x} + C.$$

In general, we have $\int e^{kx}dx = \dfrac{1}{k}e^{kx} + C$.

Since $a^x = e^{(\ln a)x}$, $\int a^x dx = \int e^{(\ln a)x}dx = \dfrac{1}{\ln a}e^{(\ln a)x} + C = \dfrac{1}{\ln a}a^x + C$.

5.3.3. $\int \frac{x^5}{\sqrt{x^3+1}}dx$.

Let $u = x^3 + 1$, $u' = 3x^2$.

$$\int \frac{x^5}{\sqrt{x^3+1}}dx = \int \frac{x^5}{\sqrt{x^3+1}}\left(\frac{1}{3x^2}\right)du = \frac{1}{3}\int \frac{u-1}{\sqrt{u}}du = \frac{1}{3}\int (u^{1/2} - u^{-1/2})du$$
$$= \frac{1}{3}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C = \frac{2}{9}\sqrt{x^3+1}(x^3+1-3) + C = \frac{2}{9}\sqrt{x^3+1}(x^3-2) + C.$$

5.3.4. $\int \frac{(\ln x+1)^2}{x}dx$.

Let $u = \ln x$, $u' = 1/x$.

$$\int \frac{(\ln x+1)^2}{x}dx = \int (u+1)^2 du = \int (u^2+2u+1)du = \frac{1}{3}u^3 + u^2 + u + C$$
$$= \frac{1}{3}(\ln x)^3 + (\ln x)^2 + \ln x + C.$$

This question suggests a general form: If we want to find an integral of the form $\int \frac{f(\ln x)}{x}dx$, where $f(\ln x)$ is a function of $\ln x$, use substitution $u = \ln x$. Then $u' = \frac{1}{x}$, and

$$\int \frac{f(\ln x)}{x}dx = \int \frac{f(\ln x)}{x}x\,du = \int f(u)du.$$

5.3.5. $\int \frac{x}{\sqrt{1+x}}dx$.

Let $u = 1 + x$. Then $u' = 1$, and $x = u - 1$.

$$\int \frac{x}{\sqrt{1+x}}dx = \int \frac{u-1}{\sqrt{u}}du = \int (u^{1/2} - u^{-1/2})du = \left(\frac{2}{3}\right)u^{3/2} - 2u^{1/2} + C$$

$$= \left(\frac{2}{3}\right)(1+x)^{3/2} - 2(1+x)^{1/2} + C = \frac{2}{3}\sqrt{1+x}(1+x-3) + C = \frac{2}{3}\sqrt{1+x}(x-2) + C.$$

5.3.6. Find $\int \frac{x+2}{x^2+2x+2}dx$.

Note that the derivative of the denominator is $2x + 2 = 2(x + 1)$. Write the numerator as $x + 2 = (x + 1) + 1$, and separate the integrand:

$$\int \frac{x+2}{x^2+2x+2}dx = \int \frac{x+1}{x^2+2x+2}dx + \int \frac{1}{x^2+2x+2}dx .$$

The first integral is easy to solve by variable substitution. Let $u = x^2 + 2x + 2$. Then

$$\int \frac{x+1}{x^2+2x+2}dx = \int \frac{x+1}{x^2+2x+2}\left(\frac{1}{2(x+1)}\right)du = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln(x^2+2x+2)+C .$$

Note that we don't need the absolute-value sign because $x^2 + 2x + 2$ is always positive.

For the second integral, we complete the square in the denominator, and use variable substitution with $u = x + 1$.

$$\int \frac{1}{x^2+2x+2}dx = \int \frac{1}{(x+1)^2+1}dx = \int \frac{1}{u^2+1}du = \arctan u + C = \arctan(x+1)+C .$$

Finally, $\displaystyle\int \frac{x+2}{x^2+2x+2}dx = \frac{1}{2}\ln(x^2+2x+2)+\arctan(x+1)+C$ .

5.3.7. $\int \tan x dx$ .

Let $u = \cos x$, $u' = -\sin x$. Then

$$\int \tan x dx = \int \frac{\sin x}{\cos x}dx = \int \left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{-\sin x}\right)du = -\int \frac{1}{u}du = -\ln|\cos x|+C .$$

## *2. Variable Substitution Used for Definite Integrals*

The technique of variable substitution can also be used directly to definite integrals. The differences are

(a) the limits have to be changed in step 2, and

(b) you don't have to go back to variable $x$ in step 5.

Suppose we want to evaluate definite integral $\int_a^b f(x)dx$ .

Step 1. Define an *intermediate variable* $u = g(x)$.

Step 2. Multiply the integrand by the reciprocal of $g'(x)$ and change $dx$ to $du$. Change the limits of integration to $g(a)$ and $g(b)$.

Step 3. Convert $\dfrac{f(x)}{g'(x)}$ as a function of $u$: $\dfrac{f(x)}{g'(x)} = h(u)$.

Step 4. If $H(u)$ is an antiderivative of $h(u)$, then $\int_{g(a)}^{g(b)} h(u)du = \left[H(u)\right]_{u=g(a)}^{g(b)}$.

This procedure can be summarized as

$\int_a^b f(x)dx = \int_{g(a)}^{g(b)} f(x)\dfrac{1}{g'(x)}du = \int_{g(a)}^{g(b)} h(u)du = \left[H(u)\right]_{u=g(a)}^{g(b)}$, where $u = g(x)$, $h(u) = \dfrac{f(x)}{g'(x)}$, and $H(u)$ is an antiderivative of $h(u)$.

*Examples*

5.3.8. $\int_0^1 x\sqrt{4-3x}dx$ .

Let $u = 4 - 3x$. Then $u' = -3$, and $x = \dfrac{1}{3}(4 - u)$.

$$\int_0^1 x\sqrt{4-3x}dx = \int_4^1 x\sqrt{4-3x}\left(\frac{1}{-3}\right)du = -\frac{1}{3}\int_4^1 \frac{1}{3}(4-u)\sqrt{u}du = -\frac{1}{9}\int_4^1 (4u^{1/2} - u^{3/2})du$$
$$= -\frac{1}{9}\left[\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_{u=4}^1 = \frac{94}{135}.$$

5.3.9. $\int_1^e \dfrac{(\ln x)^2}{x}dx$ .

Let $u = \ln x$. Then $u' = \dfrac{1}{x}$.

$$\int_1^e \frac{(\ln x)^2}{x}dx = \int_0^1 \frac{(\ln x)^2}{x}xdu = \int_0^1 (\ln x)^2 du = \int_0^1 u^2 du = \left[\frac{u^3}{3}\right]_{u=0}^1 = \frac{1}{3}.$$

5.3.10. $\int_0^{3/2} \dfrac{1}{\sqrt{9-x^2}}dx$ .

Let $u = x / 3$, $u' = 1 / 3$.

$$\int_0^{3/2} \frac{1}{\sqrt{9-x^2}}\,dx = \int_0^{3/2} \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx = \int_0^{1/2} \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}(3)du = \int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\,du = \left[\arcsin u\right]_{u=0}^{1/2} = \frac{\pi}{6}.$$

Using variable substitution for definite integrals, we have two important formulas for odd and even functions:

If $f(x)$ is an odd function, then, using $u = -x$,

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = \int_a^0 \frac{f(x)}{-1}du + \int_0^a f(x)dx$$
$$= \int_a^0 f(u)du + \int_0^a f(x)dx = -\int_0^a f(u)d(u) + \int_0^a f(x)dx = 0.$$

If $f(x)$ is an even function, then, using $u = -x$,

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = \int_a^0 \frac{f(x)}{-1}du + \int_0^a f(x)dx$$
$$= \int_a^0 -f(u)du + \int_0^a f(x)dx = \int_0^a f(u)d(u) + \int_0^a f(x)dx = 2\int_0^a f(x)dx.$$

# CHAPTER 7. TECHNIQUES OF INTEGRATION

## § 7.1. INTEGRATION BY PARTS

The method of *integration by parts* is used to find the integral of a product of two functions. This method is powerful but it can only be used to solve a relatively small group of questions. Use this method only for questions similar to examples that you see in class. **Never abuse this method!**

Suppose the integrand is a product of two functions. We denote one factor as function $u$, and let the other factor be the derivative of another function $v$, i.e., $v'$. Find the derivative of $u$, i.e., $u'$, and find an antiderivative of $v'$. i.e., $v$. Then use the formula

$$\int uv'dx = uv - \int u'vdx .$$

Because we want to find the derivative of $u$ and an antiderivative of $v'$, $u$ should have a simple derivative and $v'$ should be easy to integrate.

In some questions, the integrand is regarded as the product of a function $u$ and $v' = 1$. Then $v = x$, and

$$\int udx = xu - \int xu'dx .$$

*Examples*

7.1.1. $\int xe^{-x}dx$.

Let $u = x$, $v' = e^{-x}$. Then $u' = 1$, $v = -e^{-x}$.

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C .$$

7.1.2. $\int x\cos xdx$.

Let $u = x$, $v' = \cos x$. Then $u' = 1$, $v = \sin x$.

$$\int x\cos x dx = x\sin x - \int \sin x dx = x\sin x + \cos x + C.$$

7.1.3. $\int \sqrt{x}\ln x dx$.

Let $u = \ln x$, $v' = \sqrt{x}$. Then $u' = \dfrac{1}{x}$, $v = \dfrac{2}{3}x^{3/2}$.

$$\int \sqrt{x}\ln x dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C.$$

In general, an integral of the type $\int x^k \ln x dx$, $k \neq -1$, can be solved by integration by parts with $u = \ln x$ and $v' = x^k$:

$$\int x^k \ln x dx = \frac{1}{k+1}x^{k+1}\ln x - \int \left(\frac{1}{k+1}x^{k+1}\right)\left(\frac{1}{x}\right)dx = \frac{1}{k+1}x^{k+1}\ln x - \frac{1}{k+1}\int x^k dx$$
$$= \frac{1}{k+1}x^{k+1}\ln x - \frac{1}{(k+1)^2}x^{k+1} + C.$$

You don't have to memorize this formula, but you should know how to use this method to solve questions like $\int \dfrac{\ln x}{\sqrt{x}}dx$, $\int x\ln x dx$, $\int \dfrac{\ln x}{x^2}dx$, or $\int x^2 \ln x dx$.

Sometimes, we may have to use integration by parts more than once.

7.1.4. $\int x^2 e^{2x} dx$.

Let $u = x^2$, $v' = e^{2x}$. Then $u' = 2x$, $v = \dfrac{1}{2}e^{2x}$.

$$\int x^2 e^{2x}dx = \frac{1}{2}x^2e^{2x} - \int 2x\left(\frac{1}{2}e^{2x}\right)dx = \frac{1}{2}x^2e^{2x} - \int xe^{2x}dx.$$

In the integral on the right-hand side, use integration by parts again. Let $u = x$, $v' = e^{2x}$. Then $u' = 1$, $v = \dfrac{1}{2}e^{2x}$.

$$\frac{1}{2}x^2e^{2x} - \int xe^{2x}dx = \frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx\right) = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$$
$$= \frac{1}{4}(2x^2 - 2x + 1)e^{2x} + C.$$

The following examples use $v' = 1$:

7.1.5. $\int \ln x dx$.

Let $u = \ln x$, $v' = 1$. Then $u' = \frac{1}{x}$, $v = x$.

$$\int \ln x dx = x \ln x - \int dx = x \ln x - x + C.$$

7.1.6. $\int \arcsin x dx$.

Let $u = \arcsin x$, $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$, $v = x$.

$$\int \arcsin x dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = x \arcsin x + \sqrt{1-x^2} + C.$$

Note that, with variable substitution $t = 1 - x^2$, $t' = -2x$,

$$\int \frac{x}{\sqrt{1-x^2}} dx = \int \frac{x}{\sqrt{1-x^2}} \left(\frac{1}{-2x}\right) dt = -\frac{1}{2} \int \frac{1}{\sqrt{t}} dt = -\sqrt{t} + C = -\sqrt{1-x^2}.$$

7.1.7. $\int (\ln x)^2 dx$.

Let $u = (\ln x)^2$, and $v' = 1$. Then $u' = \frac{2\ln x}{x}$ and $v = x$.

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x dx = x(\ln x)^2 - 2x \ln x + 2x + C.$$

Some questions use integration by parts with variable substitution:

7.1.8. $\int x^3 \cos(x^2) dx$.

Let $u = x^2$. Then $u' = 2x$. $\int x^3 \cos(x^2) dx = \int x^3 \cos(x^2) \left(\frac{1}{2x}\right) du = \frac{1}{2} \int u \cos u du$.

With integration by parts as in Example 7.1.2, we have

$\int x^3\cos(x^2)dx = \frac{1}{2}(x^2\sin(x^2)+\cos(x^2)) + C.$

7.1.9. $\int e^{-\sqrt{x}}dx$.

Let $u = \sqrt{x}$. Then $u' = \frac{1}{2\sqrt{x}}$. $\int e^{-\sqrt{x}}dx = \int e^{-\sqrt{x}}\left(2\sqrt{x}\right)du = 2\int ue^{-u}du$.

By integration by part as in example 7.1.1, we have $\int e^{-\sqrt{x}}dx = -2e^{-\sqrt{x}}(\sqrt{x}+1)+C$

The following question uses integration by parts in a recursive way:

7.1.10. $\int e^x\sin xdx$.

Use integration by parts with $u = \sin x$, $v' = e^x$. Then $u' = \cos x$, $v = e^x$, and

$\int e^x\sin xdx = e^x\sin x - \int e^x\cos xdx$.

In the second integral, use integration by parts again with $u = \cos x$, $v' = e^x$. Then $u' = -\sin x$, $v = e^x$. Now we have

$\int e^x\sin xdx = e^x\sin x - \left(e^x\cos x - \int e^x(-\sin x)dx\right) = e^x(\sin x - \cos x) - \int e^x\sin xdx$.

Add $\int e^x\sin xdx$ to both sides of this equation. Note that $\int e^x\sin xdx - \int e^x\sin xdx$ is a constant $C$. Then we have

$2\int e^x\sin xdx = e^x(\sin x - \cos x) + C$.

Finally, $\int e^x\sin xdx = \frac{1}{2}e^x(\sin x - \cos x) + C$.

*Integration by parts used for definite integrals*

The formula is $\int_a^b uv'dx = \left[uv\right]_{x=a}^b - \int_a^b u'vdx$.

*Examples*

7.1.11. $\int_0^{\pi/4} \frac{x}{\cos^2 x} dx$.

Let $u = x$ and $v' = \frac{1}{\cos^2 x}$. Then $u' = 1$, and $v = \tan x$.

$$\int_0^{\pi/4} \frac{x}{\cos^2 x} dx = \left[x \tan x\right]_{x=0}^{\pi/4} - \int_0^{\pi/4} \tan x dx = \frac{\pi}{4} + [\ln|\cos x|]_{x=0}^{\pi/4} = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

7.1.12. $\int_0^1 x^2 \arctan x dx$.

Let $u = \arctan x$ and $v' = x^2$. Then $u' = \frac{1}{1+x^2}$ and $v = \frac{1}{3}x^3$.

$$\int_0^1 x^2 \arctan x dx = \left[\frac{1}{3}x^3 \arctan x\right]_{x=0}^{1} - \frac{1}{3}\int_0^1 \frac{x^3}{1+x^2} dx.$$

Use variable substitution $u = 1 + x^2$, $u' = 2x$.

$$\int_0^1 x^2 \arctan x dx = \frac{\pi}{12} - \frac{1}{6}\int \frac{u-1}{u} du == \frac{\pi}{12} - \frac{1}{6}\left[u + \ln u\right]_{u=1}^{2} = \frac{\pi}{12} - \frac{1}{6}(1 - \ln 2).$$

## § 7.2. TRIGONOMETRIC INTEGRALS

In this section, we look at three types of questions:

Type 1. $\int \sin^m x \cos^n x \, dx$.

*Solving strategy*:

(a) When $m = 2k + 1$ is odd, ($n$ may be any integer including 0), use $u = \cos x$. Then

$$\int \sin^{2k+1} x \cos^n x dx = -\int \sin^{2k} x \cos^n x du = -\int (1-u^2)^k u^n du.$$

(b) When $n = 2k + 1$ is odd, ($m$ may be any integer including 0), use $u = \overset{\text{sin}}{\cancel{\cos}}\, x$. Then

$$\int \sin^m x \cos^{2k+1} x dx = \int \sin^m x \cos^{2k} x du = \int u^m (1-u^2)^k du.$$

(c) If both odd, you can use either way.

(d) If both even, use formulas $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$, $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$, and/or $\sin x \cos x = \frac{1}{2}\sin(2x)$, and let $u = 2x$.

(There is a recursive formula to reduce the integral of $\sin^{2m} x \cos^{2n} x$ to an integral of $\sin^{2m-2} x \cos^{2n} x$ or an integral of $\sin^{2m} x \cos^{2n-2} x$. This is out of the scope of this course).

*Examples*

7.2.1 $\int \sin^3 x \cos^2 x dx$.

Let $u = \cos x$. Then $u' = -\sin x$

$$\int \sin^3 x \cos^2 x dx = \int \sin^3 x \cos^2 x\left(-\frac{1}{\sin x}\right)du = -\int \sin^2 x \cos^2 x du = -\int (1-u^2)u^2 du$$
$$= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C.$$

7.2.2. $\int_0^{\pi/2} \cos^2 x dx$.

Let $u = 2x$.

$$\int_0^{\pi/2} \cos^2 x dx = \frac{1}{2}\int_0^{\pi/2}(1+\cos(2x))dx = \frac{1}{4}\int_0^{\pi}(1+\cos u)du = \frac{1}{4}\left[u + \sin u\right]_{u=0}^{\pi} = \frac{\pi}{4}.$$

7.2.3. $\int_0^{\pi/2} \cos^3 x dx$.

Let $u = \sin x$. Then $u' = \cos x$.

$$\int_0^{\pi/2} \cos^3 x dx = \int_0^1 \cos^3 x\left(\frac{1}{\cos x}\right)du = \int_0^1 (1-\sin^2 x)du = \int_0^1 (1-u^2)du = \left[u - \frac{1}{3}u^3\right]_{u=0}^1 = \frac{2}{3}.$$

7.2.4. Evaluate $I = \int_0^{\pi/4} \sin^2 x \cos^4 x dx$.

$$I = \int_0^{\pi/4} \sin^2 x \cos^4 x dx = \frac{1}{8}\int_0^{\pi/4}(1-\cos(2x))(1+\cos(2x))^2 dx$$
$$= \frac{1}{8}\int_0^{\pi/4}(1+\cos(2x)-\cos^2(2x)-\cos^3(2x))dx.$$

With $u = 2x$,

$$I = \frac{1}{16}\int_0^{\pi/2}(1+\cos u-\cos^2 u-\cos^3 u)du = \frac{1}{16}\left(\frac{\pi}{2}+\int_0^{\pi/2}\cos u du-\int_0^{\pi/2}\cos^2 u du-\int_0^{\pi/2}\cos^3 u du\right)$$

$$= \frac{1}{16}\left(\frac{\pi}{2}+1-\frac{\pi}{4}-\frac{2}{3}\right) = \frac{\pi}{64}+\frac{1}{48}.$$

This question can be solved in another way. Use the formula $\sin x \cos x = \frac{1}{2}\sin(2x)$, and $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$. By substitution $u = 2x$,

$$I = \int_0^{\pi/4}\sin^2 x\cos^4 x dx = \frac{1}{8}\int_0^{\pi/4}\sin^2(2x)(1+\cos(2x))dx = \frac{1}{16}\int_0^{\pi/2}(\sin^2 u+\sin^2 u\cos u)du$$

$$= \frac{1}{16}\left(\int_0^{\pi/2}\sin^2 u du+\int_0^{\pi/2}\sin^2 u\cos u du\right).$$

Let $v = 2u$.

$$\int_0^{\pi/2}\sin^2 u du = \frac{1}{2}\int_0^{\pi/2}(1-\cos(2u))du = \frac{1}{4}\int_0^{\pi}(1-\cos v)dv = \frac{1}{4}\left[v-\sin v\right]_{v=0}^{\pi} = \frac{\pi}{4}.$$

Let $v = \sin u$.

$$\int_0^{\pi/2}\sin^2 u\cos u du = \int_0^1 v^2 dv = \frac{1}{3}.$$

Hence, $I = \frac{1}{16}\left(\frac{\pi}{4}+\frac{1}{3}\right) = \frac{\pi}{64}+\frac{1}{48}$.

Type 2. $\int \tan^m x\sec^n x\,dx$.

Note that $(\tan x)' = \sec^2 x$, $(\sec x)' = \tan x \sec x$, and $1 + \tan^2 x = \sec^2 x$. Then $\tan^2 x = \sec^2 x - 1$.

Solving strategy: When $m = 2k + 1 \geq 1$ is odd, use $u = \sec x$, $u' = \tan x \sec x$:

$$\int \tan^{2k+1} x\sec^n x dx = \int \tan^{2k} x\sec^{n-1} x du = \int (u^2-1)^k u^{n-1} du.$$

When $n = 2k \geq 2$, is even, use $u = \tan x$, $u' = \sec^2 x$:

$$\int \tan^m x \sec^{2k} x dx = \int \tan^m x \sec^{2(k-1)} x du = \int u^m (1+u^2)^{k-1} du .$$

If both condition is true, either way can be used; if neither condition is true, simple substitution does not work, and we have to consider other methods.

*Examples*

7.2.5. $\int \tan^3 x \sec x dx$ .

Use $u = \sec x$. $u' = \tan x \sec x$.

$$\int \tan^3 x \sec x dx = \int \tan^2 x du = \int (u^2 - 1) du = \frac{1}{3}\sec^3 x - \sec x + C .$$

7.2.6. $\int \tan^2 x \sec^4 x dx$ .

Use $u = \tan x$. $u' = \sec^2 x$.

$$\int \tan^2 x \sec^4 x dx = \int \tan^2 x \sec^2 x du = \int u^2 (1+u^2) du = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

Type 3. $\int \sin mx \sin nx\, dx, \int \cos mx \cos nx\, dx, \int \sin mx \cos nx\, dx$ .

Use formulas:

$$\sin mx \sin nx = \frac{1}{2}(\cos(m-n)x - \cos(m+n)x) ,$$

$$\cos mx \cos nx = \frac{1}{2}(\cos(m-n)x + \cos(m+n)x) ,$$

$$\sin mx \cos nx = \frac{1}{2}(\sin(m+n)x + \sin(m-n)x) .$$

*Example*

7.2.7. $\int \sin 2x \sin 3x dx = \frac{1}{2}\int (\cos x - \cos(5x)) dx = \frac{1}{2}\left( \sin x - \frac{1}{5}\sin(5x) \right) + C$ .

7.2.8. $\int_0^{\pi/2} \sin(2x)\cos(3x)dx = \frac{1}{2}\int_0^{\pi/2} (\sin(5x) - \sin x)dx$

$= \frac{1}{2}\left[-\frac{1}{5}\cos(5x) + \cos x\right]_{x=0}^{\pi/2} = \frac{1}{2}\left(0 - \left(-\frac{1}{5} + 1\right)\right) = -\frac{2}{5}$.

## § 7.3. TRIGONOMETRIC SUBSTITUTION

Trigonometric substitution is different from the substitution method in section 7.1. (This is a kind of so-called Substitution of Type II, whereas the substitution in section 7.1 is of Type I): In section 7.1, the substitution uses an intermediate variable $u = g(x)$, in substitution of Type II, $x$ is a variable of a new variable $u$: $x = g(u)$. Then

$$\int f(x)dx = \int f(x)g'(u)du = \int f(g(u))g'(u)du = \int h(u)du = H(u) + C = H(g^{-1}(x)) + C.$$

In step 2, $dx$ is replaced by $g'(u)du$. Then $h(u) = f(g(u))g'(u)$ and $H(u)$ is an antiderivative of $h(u)$. Finally, use the inverse $g^{-1}(x)$ to go back to variable $x$. Therefore, $g(u)$ must be a one-to-one function. If the integral is a definite integral, we have to change the limits from values of $x$ to values of $u$.

Some integrals involving $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$ may be solved by this method.

(a) Questions involving $a^2 - x^2$ may be solved by substitution $x = a \sin u$, where $a > 0$, and $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$.

Then $a^2 - x^2 = a^2(1 - \sin^2 u) = a^2 \cos^2 u$. Note that $\cos u \ge 0$ in this interval. Hence $\sqrt{a^2 - x^2} = a \cos u$.

(b) Questions involving $a^2 + x^2$ may be solved by substitution $x = a \tan u$, where $a > 0$, and $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

Then $a^2 + x^2 = a^2(1 + \tan^2 u) = a^2 \sec^2 u$. Note that $\sec u \ge 0$ in this interval. Hence $\sqrt{a^2 + x^2} = a \sec u$.

(c) Questions involving $x^2 - a^2$ may be solved by substitution $x = a \sec u$, where $a > 0$, and $0 \le u < \frac{\pi}{2}$ or $\pi \le u < \frac{3\pi}{2}$.

Then $x^2 - a^2 = a^2(\sec^2 u - 1) = a^2 \tan^2 u$. Note that $\tan u \ge 0$ in these intervals. Hence $\sqrt{x^2 - a^2} = a \tan u$.

*Examples*

7.3.1. $\int_0^a \sqrt{a^2 - x^2}\,dx$.

Let $x = a \sin u$, $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then $x' = a \cos u$ and $\sqrt{a^2 - x^2} = a \cos u$. Note that $\cos u \ge 0$ in this interval. When $x = 0$, $u = \arcsin 0 = 0$; when $x = a$, $u = \arcsin 1 = \frac{\pi}{2}$.

$$\int_0^a \sqrt{a^2 - x^2}\,dx = a^2 \int_0^{\pi/2} \cos^2 u\,du = \frac{a^2}{2}\int_0^{\pi/2} (1+\cos(2u))\,du = \frac{\pi a^2}{4} + \int_0^{\pi/2} \cos(2u)\,du = \frac{\pi a^2}{4}.$$

Note that, with variable substitution $v = 2u$, $\int_0^{\pi/2} \cos(2u)\,du = \frac{1}{2}\int_0^{\pi} \cos v\,dv = \frac{1}{2}(\sin\pi - \sin 0) = 0$.

7.3.2. $\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$.

Let $x = \sin u$, $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then $x' = \cos u$ and $\sqrt{1-x^2} = \cos u \ge 0$. When $x = 0$, $u = \arcsin 0 = 0$; when $x = \frac{1}{2}$, $u = \arcsin \frac{1}{2} = \frac{\pi}{6}$.

$$\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} \frac{\sin^2 u}{\cos u}(\cos u)\,du = \int_0^{\pi/6} \sin^2 u\,du = \frac{1}{2}\int_0^{\pi/6} (1-\cos(2u))\,du$$

$$= \frac{1}{2}\left[u - \frac{1}{2}\sin(2u)\right]_{u=0}^{\pi/6} = \frac{1}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) \approx 0.045293.$$

7.3.3. $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Let $x = 2\tan u$, $-\frac{\pi}{2} < u < \frac{\pi}{2}$. $x' = \frac{2}{\cos^2 u}$. $\frac{1}{\sqrt{x^2+4}} = \frac{1}{2\sqrt{\tan^2+1}} = \frac{\cos u}{2} \ge 0$.

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \left(\frac{1}{4\tan^2 u}\right)\left(\frac{\cos u}{2}\right)\left(\frac{2}{\cos^2 u}\right)du = \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\,du = -\frac{1}{4\sin u} + C.$$

Since $\tan u = x/2$, $\sin u = \dfrac{x/2}{\sqrt{1+(x/2)^2}} = \dfrac{x}{\sqrt{x^2+4}}$. Note that in interval $-\dfrac{\pi}{2} < u < \dfrac{\pi}{2}$, $\sin u$ and $\tan u$ are of the same sign.

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = -\frac{\sqrt{x^2+4}}{4x} + C.$$

7.3.4. $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\,dx$.

Let $x = \tan u$, $-\dfrac{\pi}{2} < u < \dfrac{\pi}{2}$. Then $x' = \sec^2 u$. When $x = 0$, $u = \arctan 0 = 0$; when $x = 1$, $u = \arctan 1 = \dfrac{\pi}{4}$.

$$\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \int_0^{\pi/4} \frac{1}{(1+\tan^2 u)^2}(\sec^2 u)du = \int_0^{\pi/4} \frac{\sec^2 u}{\sec^4 u}\,du = \int_0^{\pi/4} \cos^2 u\,du$$
$$= \frac{1}{2}\int_0^{\pi/4}(1+\cos(2u))du = \left[\frac{1}{2}u + \frac{1}{4}\sin(2u)\right]_{u=0}^{\pi/4} = \frac{\pi}{8} + \frac{1}{4}.$$

7.3.5. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$.

Let $x = \sec u$, $0 \le u < \dfrac{\pi}{2}$ or $\pi \le u < \dfrac{3\pi}{2}$. Then $x' = \dfrac{\sin u}{\cos^2 u}$, and $\sqrt{x^2-1} = \tan u$. Note that $\tan u \ge 0$ in these intervals.

$$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int \cos^2 u\left(\frac{\cos u}{\sin u}\right)\left(\frac{\sin u}{\cos^2 u}\right)du = \int \cos u\,du = \sin u + C = \frac{\sqrt{x^2-1}}{x} + C.$$

Note that some questions involving $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$, or $\sqrt{x^2-a^2}$ can also be solved by variable substitution:

Use $u = x^2 - 1$, $u' = 2x$. We have

$$\int \frac{x^3}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\int \frac{u+1}{\sqrt{u}}\,du = \frac{1}{2}\left(\frac{2}{3}u\sqrt{u} + 2\sqrt{u}\right) = \frac{1}{3}(x^2-1)^{3/2} + (x^2-1)^{1/2} + C = \frac{\sqrt{x^2-1}}{3}(x^2+2) + C.$$

You don't need to use trigonometric substitution for these questions.

## § 7.4. INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTION

*Partial Fractions*

The method of partial fraction is used to find the integral of proper rational functions. If the integrand is an improper rational function, use long division to write it as the sum of a polynomial and a proper rational function.

Let $f(x) = \dfrac{M(x)}{N(x)}$ be a proper rational function. Factorize the denominator $N(x)$ into a product of linear factors and irreducible quadratic factors. Recall that a quadratic polynomial $ax^2 + bx + c$ is *irreducible* if $b^2 - 4ac < 0$. An irreducible quadratic polynomial cannot be factorized further to be a product of two linear factors with real coefficients.

If $(px - q)^n$ is a factor of $N(x)$, construct a sum of *partial fractions* $\dfrac{A_1}{px-q} + \dfrac{A_2}{(px-q)^2}$ $+ \ldots + \dfrac{A_n}{(px-q)^n}$, where $A_1, A_2, \ldots, A_n$ are undetermined constants.

If $(ax^2 + bx + c)^n$ is a factor of $N(x)$, where $ax^2 + bx + c$ is an irreducible quadratic polynomial, construct a sum of *partial fractions*: $\dfrac{B_1x + C_1}{ax^2+bx+c} + \dfrac{B_2x+C_2}{(ax^2+bx+c)^2} + \ldots + \dfrac{B_nx+C_n}{(ax^2+bx+c)^n}$, where $B_1, C_1, B_2, C_2, \ldots, B_n, C_n$ are undetermined constants. Equate $\dfrac{M(x)}{N(x)}$ and the sum of all partial fractions generated by the factors of $N(x)$ to determine the constants. Then the integral of the rational function is the sum of the integrals of the partial fractions.

*Integrating Basic Partial Fractions*

Ignoring a constant factor, a partial fraction has the form $\dfrac{1}{(x+b)^n}$ or $\dfrac{x+B}{(x^2+bx+c)^n}$, where the quadratic polynomial $x^2 + bx + c$ is irreducible, i.e., $b^2 - 4c < 0$.

To integrate $\dfrac{1}{(x+b)^n}$, use variable substitution $u = x + b$. Then

$\displaystyle\int \frac{1}{x+b}\,dx = \ln|x+b| + C$, and $\displaystyle\int \frac{1}{(x+b)^n}\,dx = -\frac{1}{(n-1)(x+b)^{n+1}} + C$, $n > 1$.

To integrate $\dfrac{x+B}{(x^2+bx+c)^n}$, we first eliminate the linear term in the denominator by completing the square:

Let $u = x+\dfrac{b}{2}$, $x^2 + bx + c = u^2 + \dfrac{1}{4}(4c - b^2)$. Since $4c - b^2 > 0$, let $\dfrac{1}{4}(4c - b^2) = k^2$. Then $x^2 + bx + c = u^2 + k^2$, and $x + B = u - \dfrac{b}{2} + B = u + D$, where $D = B - \dfrac{b}{2}$. Use this substitution, the second type of partial fraction becomes $\dfrac{u}{(u^2+k^2)^n}+\dfrac{D}{(u^2+k^2)^n}$.

To integrate $\dfrac{u}{(u^2+k^2)^n}$, use variable substitution $v = u^2 + k^2$. Then

$\displaystyle\int \frac{u}{u^2+k^2}du=\frac{1}{2}\ln(u^2+k^2)+C$, when $n = 1$, and $\displaystyle\int \frac{u}{(u^2+k^2)^n}du=\frac{1}{2}\int v^{-n}dv$ = $\displaystyle\frac{1}{2(1-n)}v^{1-n}+C=-\frac{1}{2(n-1)(u^2+k^2)^{n-1}}+C$, where $n > 1$.

To integrate $\dfrac{D}{(u^2+k^2)^n}$, we have $\displaystyle\int \frac{1}{u^2+k^2}du=\frac{1}{k}\arctan\frac{u}{k}+C$, when $n = 1$.

When $n > 1$, use trigonometric substitution, $u = k \tan v$, then $u^2 + k^2 = k^2 \sec^2 v$, and $u' = k \sec^2 v$. The integral becomes

$$\int \frac{1}{(u^2+k^2)^n}du=\int \frac{\sec^2 v}{k^{2n}\sec^{2n} v}dv=\frac{1}{k^{2n}}\int \cos^{2(n-1)} v dv.$$

The last integral is solved by trigonometric integration, as discussed in Section 7.2. For instance,

$$\int \cos^4 x dx=\frac{1}{4}\int \left(1+\cos(2x)\right)^2 dx=\frac{1}{4}\left(\int dx+2\int \cos(2x)dx+\int \cos^2(2x)dx\right)$$
$$=\frac{1}{4}\left(x+\sin(2x)+\frac{1}{2}\int (1+\cos(4x))dx\right)=\frac{1}{4}\left(x+\sin(2x)+\frac{1}{2}x+\frac{1}{8}\sin(4x)\right)+C$$
$$\frac{1}{32}\left(12x+8\sin(2x)+\sin(4x)\right)+C.$$

*Examples*

7.4.1. $\displaystyle\int \frac{4x^3-6x}{2x^2+3x-2}dx$.

Use long division. We have $4x^3 - 6x = (2x - 3)(2x^2 + 3x - 2) + (7x - 6)$. Hence,

$$\frac{4x^3 - 6x}{2x^2 + 3x - 2} = 2x - 3 + \frac{7x - 6}{2x^2 + 3x - 2}.$$

Use partial fraction:

$$\frac{7x - 6}{(2x - 1)(x + 2)} = \frac{A}{2x - 1} + \frac{B}{x + 2} = \frac{A(x + 2) + B(2x - 1)}{(2x - 1)(x + 2)}.$$

Then $7x - 6 = A(x + 2) + B(2x - 1)$.

Let $x = 1 / 2$. $-5 / 2 = (5 / 2)A$. $A = -1$. Let $x = -2$, $-20 = -5B$, $B = 4$.

$$\int \frac{4x^3 - 6x}{2x^2 + 3x - 2} dx = \int (2x - 3) dx - \int \frac{1}{2x - 1} dx + 4\int \frac{1}{x + 2} dx$$
$$= x^2 - 3x - \frac{1}{2}\ln|2x - 1| + 4\ln|x + 2| + C.$$

7.4.2. $\int \frac{x + 1}{x^3 + 2x^2 + 5x} dx$.

Factorize the denominator: $x^3 + 2x^2 + 5x = x(x^2 + 2x + 5)$.

Let $\frac{x + 1}{x(x^2 + 2x + 5)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2x + 5} = \frac{A(x^2 + 2x + 5) + (Bx + C)x}{x(x^2 + 2x + 5)}$.

Then $A(x^2 + 2x + 5) + (Bx + C)x = x + 1$, or

$(A + B)x^2 + (2A + C)x + 5A = x + 1$.

Comparing the coefficients on both sides, $A + B = 0$, $2A + C = 1$, $5A = 1$. Then $A = \frac{1}{5}$,

$B = -\frac{1}{5}$, and $C = \frac{3}{5}$.

Hence, $\int \frac{x + 1}{x(x^2 + 2x + 5)} dx = \frac{1}{5}\int \frac{1}{x} dx + \frac{1}{5}\int \frac{3 - x}{x^2 + 2x + 5} dx$.

Since $x^2 + 2x + 5 = (x + 1)^2 + 4$, let $u = x + 1$. Then $3 - x = 4 - u$.

$$\int \frac{3 - x}{x^2 + 2x + 5} dx = 2\int \frac{4 - u}{u^2 + 4} du = \int \left(\frac{4}{u^2 + 4} - \frac{u}{u^2 + 4}\right) du$$

$$= 2\arctan\frac{u}{2} - \frac{1}{2}\ln(u^2+4) + C = 2\arctan\left(\frac{x+1}{2}\right) - \frac{1}{2}\ln(x^2+2x+5) + C.$$

Finally, $\displaystyle\int \frac{x+1}{x(x^2+2x+5)}\,dx = \frac{1}{5}\ln|x| + \frac{2}{5}\arctan\left(\frac{x+1}{2}\right) - \frac{1}{10}\ln(x^2+2x+5) + C$.

7.4.3. $\displaystyle\int \frac{1}{x^2(x-2)}\,dx$.

Let $\displaystyle\frac{1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2) + B(x-2) + Cx^2}{x^2(x-2)}$.

Then $Ax(x-2) + B(x-2) + Cx^2 = 1$.

Let $x = 2$. $4C = 1$, $C = \dfrac{1}{4}$. Let $x = 0$. $-2B = 1$. $B = -\dfrac{1}{2}$.

Let $x = 1$. $-A - B + C = 1$. $A = -B + C - 1 = -\dfrac{1}{4}$.

$$\int \frac{1}{x^2(x-2)}\,dx = \frac{1}{4}\int\left(-\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x-2}\right)dx = \frac{1}{4}\left(-\ln|x| + \frac{2}{x} + \ln|x-2|\right) + C.$$

7.4.4. $\displaystyle\int \frac{x^3+x+1}{x(x^2+1)^2}\,dx$.

Let $\displaystyle\frac{x^3+x+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$.

Then $A(x^2+1)^2 + (Bx+C)(x^3+x) + (Dx+E)x = x^3 + x + 1$.

Let $x = 0$. We see that $A = 1$.

$(x^4 + 2x^2 + 1) + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex = x^3 + x + 1$,

$(1 + B)x^4 + Cx^3 + (2 + B + D)x^2 + (C + E)x + 1 = x^3 + x + 1$.

Then $B = -1$, $C = 1$, $2 + B + D = 1 + D = 0$, $D = -1$, $C + E = 1 + E = 1$, $E = 0$.

Hence,

$$\int \frac{1}{x(x^2+1)^2}\,dx = \int \left(\frac{1}{x} - \frac{x}{x^2+1} + \frac{1}{x^2+1} - \frac{x}{(x^2+1)^2}\right)dx$$
$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \arctan x + \frac{1}{2(x^2+1)} + C\,.$$

## § 7.5. Techniques of Integration

Typical type of questions:

(i) $\int \frac{f(\ln x)}{x}\,dx$. Solved by substitution $u = \ln x$.

(ii) $\int x^k f(x)dx$, where $f(x) = \ln x$ or $\arctan x$. Use integration by parts. Let $u = f(x)$ and $v' = x^k$.

*Examples*

7.5.1. $\int \frac{\sqrt{\ln x}}{x}\,dx$.

Let $u = \ln x$. Then $u' = \frac{1}{x}$, and

$$\int \frac{\sqrt{\ln x}}{x}\,dx = \int \frac{\sqrt{\ln x}}{x}\,x\,du = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\ln x)^{3/2} + C\,.$$

7.5.2. $\int x \arctan x\,dx$.

Use integration by parts. Let $u = \arctan x$, $v' = x$. Then $u' = \frac{1}{1+x^2}$ and $v = \frac{1}{2}x^2$.

$$\int x\arctan x\,dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx$$
$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\left(x - \arctan x\right) + C\,.$$

The following examples use a combination of different methods:

7.5.3. $\int \sec x dx$ .

First, $\int \sec x dx = \int \frac{1}{\cos x} dx$ . Let $u = \sin x$. Then $u' = \cos x$, and
$\int \frac{1}{\cos x} dx = \int \frac{1}{\cos^2 x} du = \int \frac{1}{1-u^2} du$ .

Use partial fraction, $\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)$. Hence,

$$\int \sec x dx = \int \frac{1}{1-u^2} du = \frac{1}{2}\left(\int \frac{1}{1-u} du + \int \frac{1}{1+u} du\right) = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + C .$$

Because $\frac{1+\sin x}{1-\sin x}$ is non-negative, we don't need the absolute-value sign. This result can be simplified further. Multiply both top and bottom by $1 + \sin x$. We have

$$\frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) = \ln\sqrt{\frac{1+\sin x}{1-\sin x}} = \ln\sqrt{\frac{(1+\sin x)^2}{\cos^2 x}} = \ln\left|\frac{1+\sin x}{\cos x}\right| = \ln|\sec x + \tan x| .$$

Hence, $\int \sec x dx = \ln|\sec x + \tan x| + C$ .

(This question can also be solved by a trickier, but easier, method. See textbook p.483).

7.5.4. $\int_0^{\pi/6} \tan^2 x \sec x dx$ .

$\int_0^{\pi/6} \tan^2 x \sec x dx = \int_0^{\pi/6} \frac{\sin^2 x}{\cos^3 x} dx$ . Use $u = \sin x$, $u' = \cos x$. Then

$\int_0^{\pi/6} \tan^2 x \sec x dx = \int_0^{\pi/6} \frac{\sin^2 x}{\cos^3 x} dx = \int_0^{\pi/6} \frac{\sin^2 x}{\cos^4 x} du = \int_0^{1/2} \frac{u^2}{(1-u^2)^2} du$ . This integral can be solved by partial fraction:

$$\int_0^{1/2} \frac{u^2}{(1-u^2)^2} du = \frac{1}{4}\left(-\int_0^{1/2} \frac{1}{1-u} du - \int_0^{1/2} \frac{1}{1+u} du + \int_0^{1/2} \frac{1}{(1-u)^2} du + \int_0^{1/2} \frac{1}{(1+u)^2} du\right)$$
$$= \frac{1}{4}\left[\ln\left|\frac{1-u}{1+u}\right| + \frac{1}{1-u} - \frac{1}{1+u}\right]_{u=0}^{1/2} = \frac{1}{4}\left(\ln\frac{1}{3} + 2 - \frac{2}{3}\right) = \frac{1}{3} - \frac{1}{4}\ln 3 .$$

## § 7.7. APPROXIMATE INTEGRATION

Since the definite integral is defined as the limit of a sequence of Riemann sums, we can use a Riemann sum as an approximation of the definite integral $\int_a^b f(x)dx$.

Let $h = \dfrac{b-a}{n}$, and $x_i = a + ih$, $i = 0, 1, 2, \ldots, n$.

If $x_i^* = x_{i-1}$, $i = 1, 2, \ldots, n$, we have the *left sum*

$L(n) = h\,(f(x_0) + f(x_1) + \ldots + f(x_{n-1}))$.

If $x_i^* = x_i$, $i = 1, \ldots, n$, we have the *right sum*

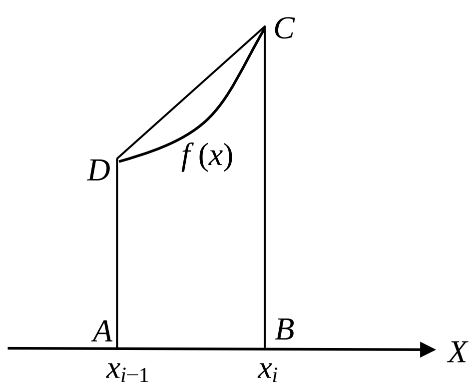
$R(n) = h\,(f(x_1) + f(x_2) + \ldots + f(x_n))$.

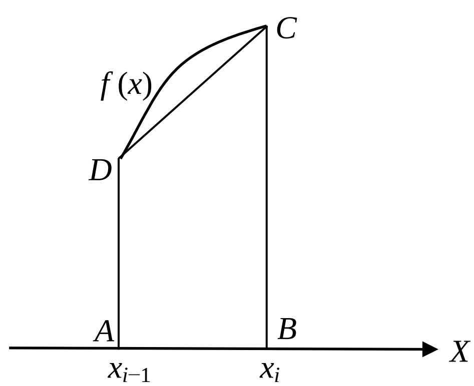
If $f(x)$ is increasing on $[a, b]$, then the right sum gives an overestimate, and the left sum gives an underestimate. If $f(x)$ is decreasing on $[a, b]$, then the right sum gives an underestimate, and the left sum gives an overestimate.

We may also take the average of the left sum and the right sum as a (probably) better approximation. This is called the *trapezoidal rule*:

$$T(n) = \frac{1}{2}(L(n) + R(n)) = \frac{1}{2}h\,(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)).$$

In an interval $[x_{i-1}, x_i]$, the trapezoidal rule uses the area of a trapezoid with the secant line joining points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ as the top side (trapezoid $ABCD$) as shown in the following figure:



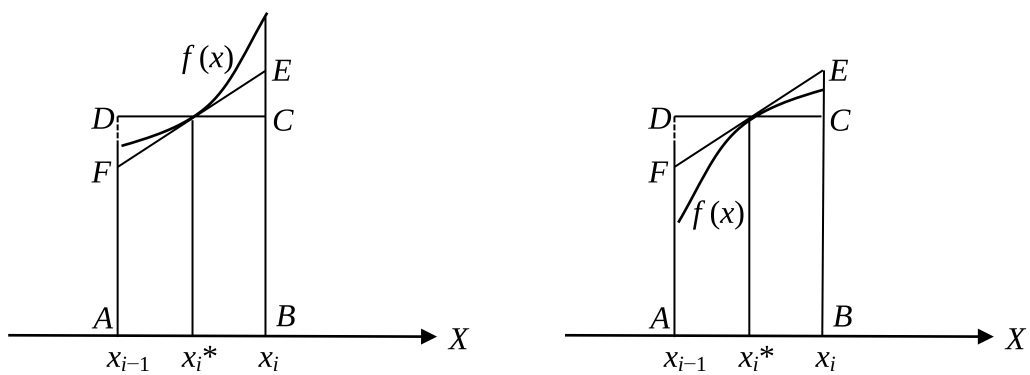



From this figure, we can see that the trapezoidal rule gives an overestimate when $f(x)$ is concave up in the interval of integration, and it gives an underestimate when $f(x)$ is concave down in the interval of integration

If we take the midpoint of each subinterval $[x_{i-1}, x_i]$, $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$, then we have the *midpoint rule*:

$M(n) = h\,(f(x_1^*) + f(x_2^*) + \ldots + f(x_n^*))$.

In an interval $[x_{i-1}, x_i]$, the midpoint rule uses the area of the rectangle of height $f(x_i^*)$ (rectangle $ABCD$) to approximate the area under the graph of $f(x)$, the area of this rectangle equals the area of the trapezoid with the tangent line of $f(x)$ at $x = x_i^*$ as the top side (trapezoid $ABEF$) as shown in the following figure:




From this figure, we can see that the midpoint Rule gives an underestimate when $f(x)$ is concave up in the interval of integration, and it gives an overestimate when $f(x)$ is concave down in the interval of integration.

There is another more accurate formula to approximate a definite integral $\int_a^b f(x)dx$, called the *Simpson's rule*. Let $n = 2m$ be an even number, and let $h = \frac{b-a}{n}$. Use mesh points $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, …, $x_n = a + nh = b$ to subdivide the interval of integration $[a, b]$ into $n$ subintervals. The Simpson's rule uses the sum

$$S(n) = \frac{1}{3}(T(m) + 2M(m))$$

$$= \frac{h}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_{2k-1}) + 2f(x_{2k}) + \ldots + 4f(x_{n-1}) + f(x_n))$$

to be an approximation of definite integral $\int_a^b f(x)dx$.

*Example* 7.1.1. Suppose we want to estimate $\int_2^4 \frac{1}{x}\,dx$ with $n = 6$. Then $h = \frac{4-2}{6} = \frac{1}{3}$.

The mesh points are $x_0 = 2$, $x_1 = 2+\frac{1}{3} = \frac{7}{3}$, $x_2 = 2+\frac{2}{3} = \frac{8}{3}$, $x_3 = 2+\frac{3}{3} = 3$, $x_4 = 2+\frac{4}{3} = \frac{10}{3}$, $x_5 = 2+\frac{5}{3} = \frac{11}{3}$, $x_6 = 2+\frac{6}{3} = 4$.

Hence,

$$R(6) = \frac{1}{3}\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}+\frac{1}{x_6}\right) \approx 0.65321$$

$$L(6) = \frac{1}{3}\left(\frac{1}{x_0}+\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}\right) \approx 0.73654$$

$$T(6) = \frac{1}{6}\left(\frac{1}{x_0}+\frac{2}{x_1}+\frac{2}{x_2}+\frac{2}{x_3}+\frac{2}{x_4}+\frac{2}{x_5}+\frac{1}{x_6}\right) \approx 0.69488$$

$$S(6) = \frac{1}{9}\left(\frac{1}{x_0}+\frac{4}{x_1}+\frac{2}{x_2}+\frac{4}{x_3}+\frac{2}{x_4}+\frac{4}{x_5}+\frac{1}{x_6}\right) \approx 0.69317$$

To use the midpoint rule, we have to find $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$, $i = 1, 2, 3, 4, 5, 6$. Then

$x_1^* = \frac{1}{2}\left(2+\frac{7}{3}\right) = \frac{13}{6}$, $x_2^* = \frac{1}{2}\left(\frac{7}{3}+\frac{8}{3}\right) = \frac{15}{6}$, $x_3^* = \frac{1}{2}\left(\frac{8}{3}+\frac{9}{3}\right) = \frac{17}{6}$, $x_4^* = \frac{1}{2}\left(\frac{9}{3}+\frac{10}{3}\right) = \frac{19}{6}$,
$x_5^* = \frac{1}{2}\left(\frac{10}{3}+\frac{11}{3}\right) = \frac{21}{6}$, $x_6^* = \frac{1}{2}\left(\frac{11}{3}+\frac{12}{3}\right) = \frac{23}{6}$.

$$M(6) = \frac{1}{3}\left(\frac{1}{x_1^*}+\frac{1}{x_2^*}+\frac{1}{x_3^*}+\frac{1}{x_4^*}+\frac{1}{x_5^*}+\frac{1}{x_6^*}\right) \approx 0.69228.$$

When $f(x)$ is monotonic on $[a, b]$, $|\,R(n) - L(n)\,|$ gives an error bound of the estimation. Since $|\,R(n) - L(n)\,| = h\,|\,f(b) - f(a)\,|$, we can use $f(b) - f(a)$ to find appropriate value of $h$ that guarantees to meet the error requirement.

*Example* 7.1.1. (continued) Since function $f(x) = \dfrac{1}{x}$ is increasing in interval [2, 4], if we want to find an approximation of $\int_2^4 \frac{1}{x}\,dx$ using the left or the right sum with an error less than $10^{-3}$, we should have $h < \left|\dfrac{1/4-1/2}{0.001}\right| = \left|\dfrac{0.25}{0.001}\right| = 250$.

# CHAPTER 4. APPLICATIONS OF DERIVATIVES

## § 4.1 - 4.3, 4.5. USE DERIVATIVES TO STUDY PROPERTIES OF FUNCTIONS

### *1. Critical Numbers of a Function*

*The Mean Value Theorem*

If $f(x)$ is defined and continuous on $[a, b]$, and it is differentiable in $(a, b)$, then there exists a value $c$ in $(a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

In other words, the tangent line at $x = c$ is parallel to the secant line joining points $(a, f(a))$ and $(b, f(b))$.

*Roll's Theorem*

Suppose $f(x)$ is defined and continuous on a closed interval $[a, b]$, it is differentiable in $(a, b)$, and $f(a) = f(b)$. Then there is $c$ in $(a, b)$ where $f'(c) = 0$.

*Critical Number*

A value $x = a$ is a critical number of a function $y = f(x)$ if (i) $a$ is in the domain of $f(x)$, and (ii) $f'(a) = 0$ or $f'(a)$ does not exist.

*Local Maximum and Local Minimum*

Let $x = c$ be a critical number of function $f(x)$ in an interval $(a, b)$, $a < c < b$. If $f(x)$ attains a local maximum or a local minimum at $x = c$, then $c$ must be a critical number of $f(x)$.

On the other hand, at a critical number $c$ of function $f(x)$, this function does not have to attain a local maximum or a local minimum.

We may use either the first derivative test or the second derivative test to determine whether this function attains a local maximum, or a local minimum, or neither, at a critical number $x = c$.

*First Derivative Test*:

Let $c$ be a critical number of $f(x)$.

If there exists an interval $(a, b)$, $a < c < b$, such that $f'(x) < 0$ when $a < x < c$, and $f'(x) > 0$ when $c < x < b$, then $f(x)$ attains a local minimum at $x = c$.

If there exists an interval $(a, b)$, $a < c < b$, such that $f'(x) > 0$ when $a < x < c$, and $f'(x) < 0$ when $c < x < b$, then $f(x)$ attains a local maximum at $x = c$.

Otherwise, this function does not attain a local maximum or local minimum at this critical number.

*Second Derivative Test*:

Let $c$ be a critical number of $f(x)$. Suppose $f''(c)$ exists.

If $f''(c) > 0$, then $f(x)$ attains a local minimum at $x = c$.

If $f''(c) < 0$, then $f(x)$ attains a local maximum at $x = c$.

If $f''(c) = 0$, this method fails.

### *2. First Derivative Analysis*

*Purpose*: Use the first derivative of a function $f(x)$ to determine the intervals where this function is increasing, and the intervals where this function is decreasing. Find the local / global maximum / minimum of this function, if any. Find the global maximum and the global minimum of this function in its entire domain.

*Steps*:

(1) Find the first derivative $f'(x)$.

(2) Find the critical numbers.

(3) The critical numbers subdivide the domain of the function into a number of subintervals. Take a particular value in each of these subintervals, and plug it into the first derivative of the function to determine the sign of the first derivative in this subinterval.

(4) Use the sign of the derivative in each of these subintervals to determine the intervals where the function is increasing or decreasing.

(5) Use the first derivative test to find the values of $x$ where this function attains a local maximum or a local minimum, if any.

(6) Find the values of the function on the ends of intervals of its domain, or study the behaviour of the function when the variable approaches an open end of a domain interval, to find the global extrema.

### *3. Second Derivative Analysis*

*Purpose*: Use the second derivative to find interval(s) where the graph of the function is concave up or concave down, and inflection points.

*Steps*:

(1) Find the second derivative $f''(x)$.

(2) Find values of $x$ that are (a) in the domain of $f(x)$, and (b) $f''(x) = 0$ or $f''(x)$ does not exist. (Note that, these numbers are NOT critical numbers! No name is assigned to these numbers.)

(3) The numbers found in step 2 subdivide the domain of the function into a number of subintervals. Take a particular value in each of these subintervals, and plug it into the second derivative of the function to determine the sign of the second derivative in this subinterval.

(4) Use the sign of the second derivative in each of these subintervals to determine the intervals where the graph of the function is concave up or concave down.

(5) Find all inflection points, if any.

### *4. Graph Sketching*

Using the properties obtained by the first and second derivative analysis, combined with the information about the horizontal / vertical asymptotes, and the coordinates of some particular points, we can sketch the graph of the function.

*Examples*

4.1.1. A function $y = f(x)$ defined and continuous for all $x \neq 0$ has the following properties:

(i) $f'(x)\begin{cases} <0, & -3<x<0 \text{ or } x>0 \\ =0, & x=-3 \\ >0, & x<-3 \\ \text{not defined,} & x=0 \end{cases}.$

(ii) $f''(x)\begin{cases} <0, & -5<x<0 \\ =0, & x=-5 \\ >0, & x<-5 \text{ or } x>0 \\ \text{not defined,} & x=0 \end{cases}.$

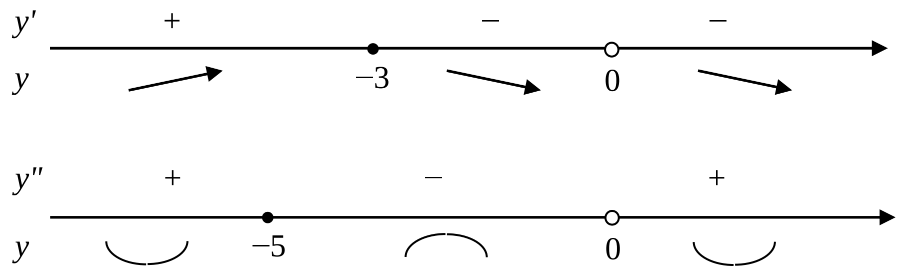
(iii) $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$;

(iv) $\lim_{x\to 0^-} f(x) = -\infty, \ \lim_{x\to 0^+} f(x) = \infty$.

(v) $f(-5) = 1, \ f(-3) = 2, \ f(1) = 1$.

The domain of this function consists of two intervals $(-\infty, 0)$ and $(0, \infty)$.

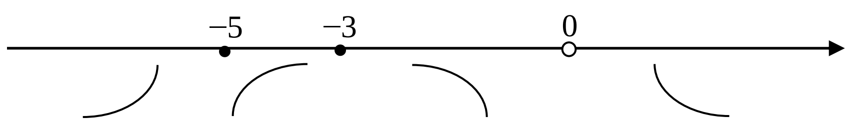
According to the conditions (i) and (ii), construct the following diagrams:



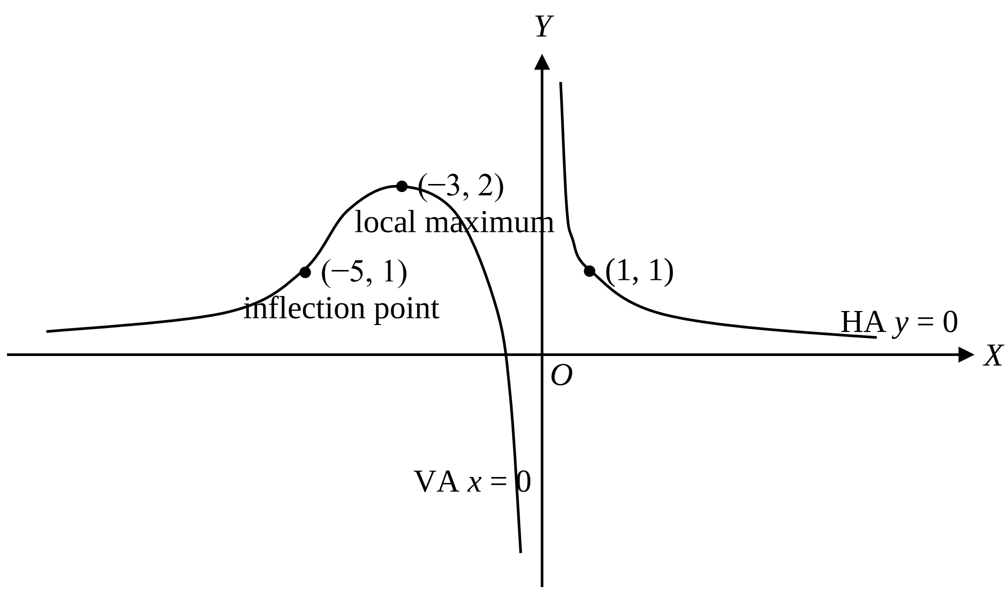

Function $f(x)$ is increasing when $x < 3$, and it is decreasing when $-3 < x < 0$ or $x > 0$. The function attains a local maximum at $x = -3$. By condition (iv), this function does not have the global maximum or global minimum.

The graph of the function $f(x)$ is concave up when $x < -5$ or $x > 0$, and it is concave down when $-5 < x < 0$. It has an inflection point at $x = -5$.

The shape of the graph in each interval looks like the following:




Condition (iii) tells us that $x = 0$ is a vertical asymptote, and condition (iv) tells us that $y = 0$ is a horizontal asymptote. By the particular values given in condition (v), we have three points on the graph: $(-5, 1)$, $(-3, 2)$ and $(1, 1)$. The graph of this function looks like the following:



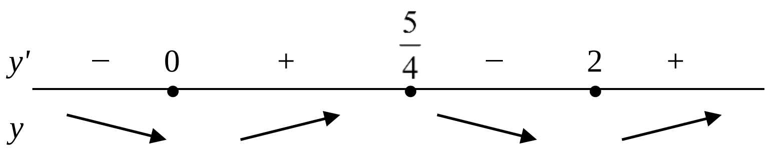

4.1.2. $y = 3x^4 - 13x^3 + 15x^2$ defined on $[-1, 3]$.

*First derivative* analysis:

$y' = 12x^3 - 39x^2 + 30x = 3x(4x^2 - 13x + 10)$.

Let $y' = 0$. $x = \dfrac{13 \pm \sqrt{169 - 160}}{8} = \dfrac{13 \pm 3}{8} = \dfrac{5}{4}, 2$.

Critical numbers: $x = 0, \dfrac{5}{4}, 2$. Construct a diagram as follows:




From the diagram, we see that $y' < 0$ when $x < 0$ or $\dfrac{5}{4} < x < 2$, and $y' > 0$ when $0 < x < \dfrac{5}{4}$ or $x > 2$. This function increases in intervals $(0, 5 / 4)$ and $(2, \infty)$, and it decreases in intervals $(-\infty, 0)$ and $(5 / 4, 2)$. Function $y$ attains two local minima at $x = 0$ and $x = 2$, and it attains a local maximum at $x = \dfrac{5}{4}$.

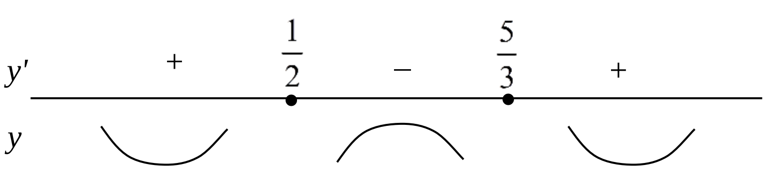
$y(0) = 0$, $y(2) = 4$, $y(5 / 4) = 1375 / 256 \approx 5.4$.

Since $y(-1) = 31$, $y(3) = 27$. This function attains its absolution maximum $y(-1) = 27$, and global minimum $y(0) = 0$.

*Second derivative analysis*:

$y'' = 36x^2 - 78x + 30$.

Let $y'' = 0$. $x = \dfrac{13 \pm \sqrt{169 - 120}}{12} = \dfrac{13 \pm 7}{12} = \dfrac{1}{2}, \dfrac{5}{3}$. Construct a diagram as follows:

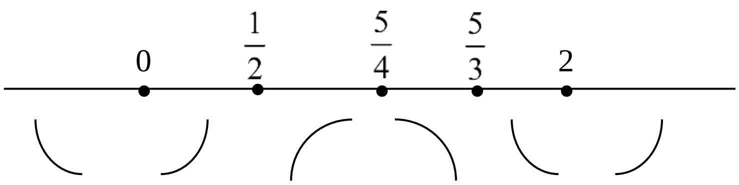



From the diagram, we see that $y'' < 0$ when $\dfrac{1}{2} < x < \dfrac{5}{3}$, and $y'' > 0$ when $x < \dfrac{1}{2}$ or $x > \dfrac{5}{3}$.

The graph of this function $y$ has inflection points at $x = \frac{1}{2}$ and $x = \frac{5}{3}$. $y\left(\frac{1}{2}\right) = \frac{37}{16} \approx 2.3$, $y\left(\frac{5}{3}\right) = \frac{125}{27} \approx 4.6$, The graph of this function is concave up in intervals $(-\infty, 1/2)$ and $(5/3, \infty)$, and it is concave down in interval $(1/2, 5/3)$. It has inflection points at $x = 1/2$, and $x = 5/3$.

*Asymptotes*: No asymptotes.

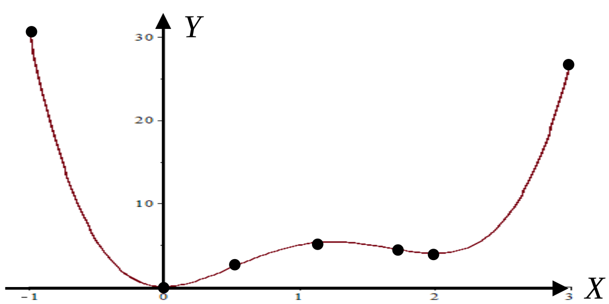
*Putting together*: Construct the following diagram, and we know the shape of the graph in each of the subintervals:




*Find some particular values of this function*:

$y(-1) = 31$, $y(0) = 0$, $y\left(\frac{1}{2}\right) = \frac{37}{16} \approx 2.3$, $y\left(\frac{5}{4}\right) = \frac{1375}{256} \approx 5.4$, $y\left(\frac{5}{3}\right) = \frac{125}{27} \approx 4.6$, $y(2) = 4$, $y(3) = 27$.

*The graph of this function looks like the following*:

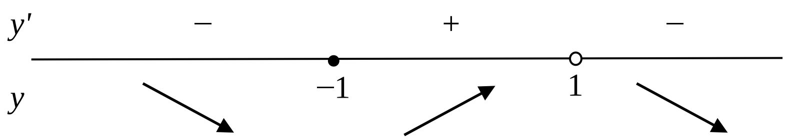



4.1.3. $f(x) = \frac{6x}{(x-1)^2}$. The domain of this function is $(-\infty, 1)$ and $(1, \infty)$.

*First derivative analysis*:

$$y' = -\frac{6(x+1)}{(x-1)^2}.$$

Let $y' = 0$. Then $x = -1$. This function has a unique critical number $x = -1$. Construct the following diagram:

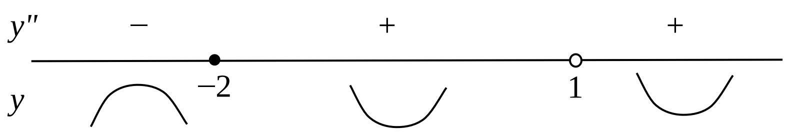



This function increases in interval $(-1, 1)$ and it decreases in intervals $(-\infty, -1)$ and $(1, \infty)$. It attains a local minimum at $x = -1$, $y(-1) = -\frac{3}{2}$.

When $x$ approaches $1^+$ or $1^-$, $y$ approaches infinity, this function does not have a global maximum. When $x$ approaches infinity or negative infinity, $y$ approaches 0. Hence, the local minimum is the global minimum.

*Second derivative analysis*:

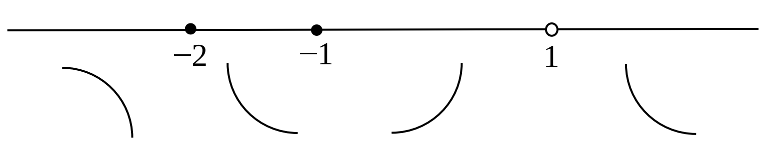
$y'' = \dfrac{12(x+2)}{(x-1)^4}$. Let $y'' = 0$. $x = -2$. Construct the following diagram:




The graph of this function is concave up in $(-1, 2)$, and $(1, \infty)$, and it is concave down in $(-\infty, -2)$. It has an inflection point at $x = -2$, $y(-2) = -4 / 3$.

*Asymptotes*: Since $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$, $y = 0$ is a horizontal asymptote. Since $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = \infty$, $x = 1$ is a vertical asymptote.

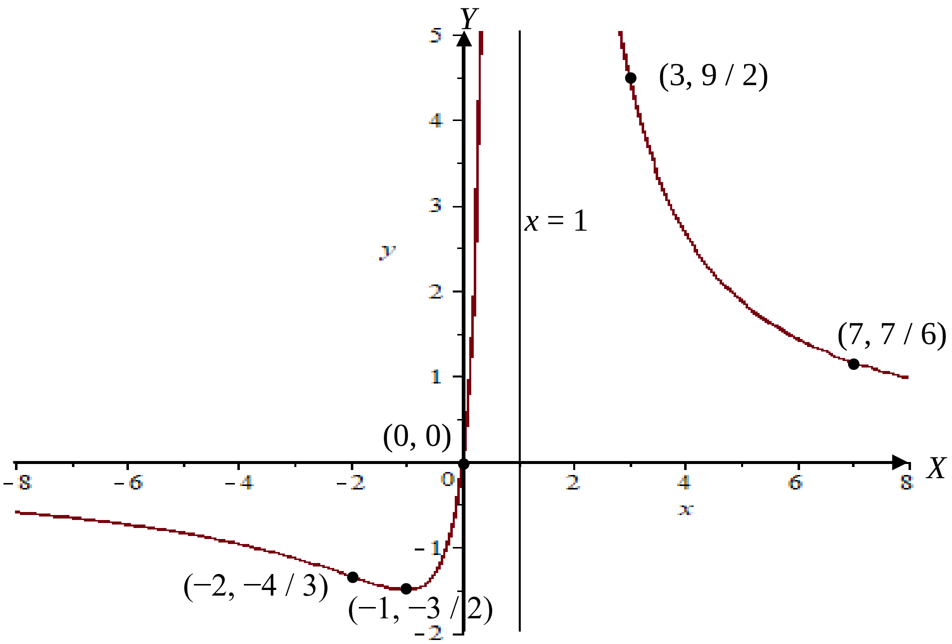
*Putting together*: Construct the following diagram to determine the shape of the graph in each subinterval:




*Find some particular values of this function*:

| | | | | | |
|---|---|---|---|---|---|
| $x$ | $-2$ | $-1$ | $0$ | $3$ | $7$ |
| $y$ | $-4 / 3$ | $-3 / 2$ | $0$ | $9 / 2$ | $7 / 6$ |

*The graph of this function looks like the following*:




4.1.4. $y = x^2e^x$.

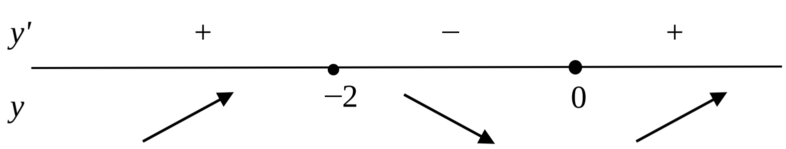
This function is defined for all real numbers.

*First derivative analysis*:

$y' = 2xe^x + x^2e^x = x(2 + x)e^x$.

Let $y' = 0$. Then we have critical numbers $x = 0$ and $x = -2$.

Construct the following diagram:



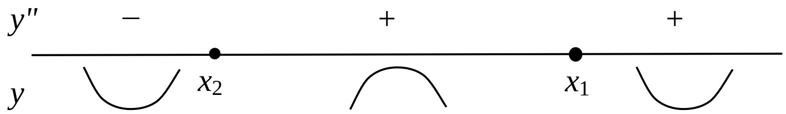

This function increases in intervals $(-\infty, -2)$ and $(0, \infty)$, and it decreases in interval $(-2, 0)$. It attains a local maximum at $x = -2$, $y(-2) = 4e^{-2} \approx 0.54$, and a local minimum at $x = 0$, $y(0) = 0$.

When $x$ approaches infinity, this function approaches infinity. This function does not have a global maximum. Since $y \geq 0$ for all $x$, the local minimum $y(0) = 0$ is the global minimum.

*Second derivative analysis*:

$y'' = (2 + 2x)e^x + (2x + x^2)e^x = (2 + 4x + x^2)e^x$.

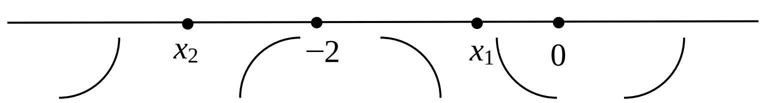
Let $y'' = 0$. We have $x = x_1 = -2 + \sqrt{2} \approx -0.59$, $x = x_2 = -2 - \sqrt{2} \approx -3.41$. Construct the following diagram:




The graph of this function is concave up in $(-\infty, x_2)$, and $(x_1, \infty)$, and it is concave down in $(x_2, x_1)$. It has inflection points at $x = x_1$ and $x = x_2$, $y(x_1) = 0.19$, and $y(x_2) \approx 0.38$.

*Asymptotes*: Since $\lim_{x \to -\infty} y = 0$, $y = 0$ is a horizontal asymptote. It does not have a vertical asymptote.

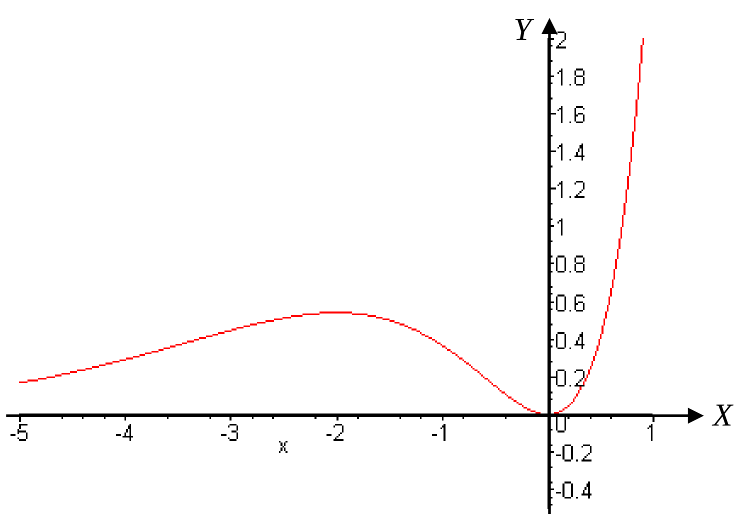
*Putting together*: Construct the following diagram to determine the shape of the graph in each subinterval:




*Find some particular values of this function*:

| | | | | |
|---|---|---|---|---|
| $x$ | $x_2 \approx 3.41$ | $-2$ | $x_1 \approx 0.59$ | 0 |
| $y$ | $\approx 0.19$ | $\approx 0.54$ | $\approx 0.38$ | 0 |

*The graph of this function looks like the following*:



4.1.5. $y = \dfrac{x^{1/5}}{x+1}$.

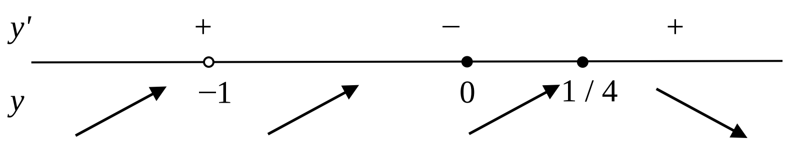
This function is defined for all $x \neq -1$.

*First derivative analysis*:

$$y' = \frac{1-4x}{5x^{4/5}(x+1)^2},$$

Let $y' = 0$. Then $x = 1 / 4$. The derivative $y'$ does not exist when $x = 0$ or $x = -1$. Since $x = -1$ is not in the domain of this function, this function has two critical numbers $x = 1 / 4$, and $x = -1$.

Construct the following diagram:

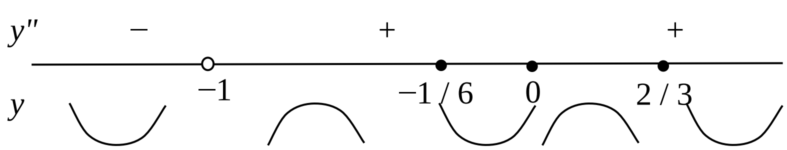



From this diagram, we see that $y' > 0$ when $x < -1$, $-1 < x < 0$, or $0 < x < 1 / 4$; $y' < 0$ when $x > 1 / 4$. This function increases in intervals $(-\infty, -1)$, $(-1, 0)$, and $(0, 1 / 4)$, and it decreases in interval $(1 / 4, \infty)$. It attains a local maximum at $x = 1 / 4$. $y(1 / 4) \approx 0.61$. Since $y'$ approaches infinity when $x$ approaches 0, $y$ has a vertical tangent line at $x = 0$.

*Second derivative analysis*:

$$y'' = \frac{2(18x^2 - 9x - 2)}{25x^{9/5}(x+1)^3}.$$

Let $y'' = 0$. $x = 2 / 3, -1 / 6$. The second derivative $y''$ does not exist when $x = -1$, and $x = 0$. Construct the following diagram:

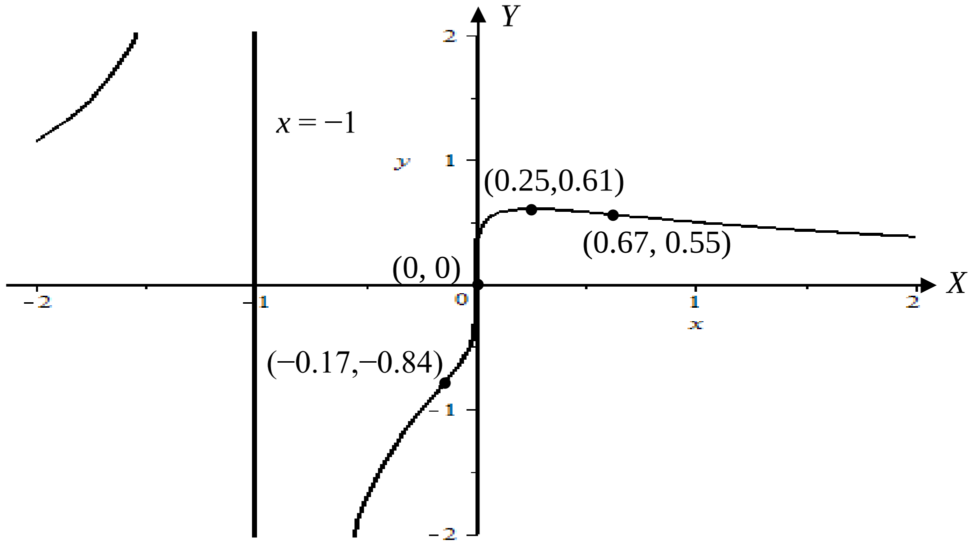



The graph of this function is concave up in intervals $(-\infty, -1)$, $(-1 / 6, 0)$, and $(2 / 3, \infty)$. It is concave down in intervals $(-1, -1 / 6)$ and $(0, 2 / 3)$. It has inflection points at $x = -1 / 6$, $x = 0$, and $x = 2 / 3$. $y(-1 / 6) \approx -0.84$, $y(0) = 0$, $y(2 / 3) \approx 0.55$.

*Asymptotes*:

Since $\lim_{x\to-\infty} y = \lim_{x\to\infty} y = 0$, $y = 0$ is a horizontal asymptote. Since $\lim_{x\to-1^-} y = \infty$, $\lim_{x\to-1^+} y = -\infty$, $x = -1$ is a vertical asymptote.

*The graph of this function looks like the following*:




## § 4.4. INDETERMINATE FORMS AND L'HOPITAL'S RULES

An *indeterminate form* is a limit $\lim_{x\to a} \frac{f(x)}{g(x)}$, where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, symbolically called a $\frac{0}{0}$ *form*, or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, symbolically called an $\frac{\infty}{\infty}$ *form*.

L'Hopital's Rule: If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}$.

***1. Use the Rule Directly***

*Examples*

4.2.1. $\lim_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{\sin x + x\cos x}{\sin x} = \lim_{x\to 0} \frac{\cos x + \cos x - x\sin x}{\cos x} = 2$.

4.2.2. $\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$.

4.2.3. $\lim_{x\to 0}\frac{x\sin x}{1-\cos x}=\lim_{x\to 0}\frac{\sin x + x\cos x}{\sin x}=\lim_{x\to 0}\frac{\cos x+\cos x - x\sin x}{\cos x}=2$.

4.2.4. Consider the limit $\lim_{x\to\infty}\frac{x^k}{e^x}$, where $k$ is any real number. If $k \le 0$, this limit is obviously zero. If $k > 0$, this is an $\frac{\infty}{\infty}$ form. Use L'Hopital's rule $\lim_{x\to\infty}\frac{x^k}{e^x}=\lim_{x\to\infty}\frac{kx^{k-1}}{e^x}$ repeatedly until the exponent of $x$ becomes non-positive. Then $\lim_{x\to\infty}\frac{x^k}{e^x}=0$ for all real number $k$.

4.2.5. Consider the limit $\lim_{x\to\infty}\frac{\ln x}{x^k}$, where $k > 0$. Then it is an $\frac{\infty}{\infty}$ form. Using L'Hopital's rule, we have $\lim_{x\to\infty}\frac{\ln x}{x^k}=\lim_{x\to\infty}\frac{1/x}{kx^{k-1}}=\lim_{x\to\infty}\frac{1}{kx^k}=0$.

***2. Product***

The limit $\lim_{x\to a} f(x)g(x)$, where $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$, is, symbolically, a $0\times\infty$ *form*.

To find this limit, write $\lim_{x\to a} f(x)g(x)=\lim_{x\to a}\frac{f(x)}{1/g(x)}$ to convert it to a $\frac{0}{0}$ form, or write $\lim_{x\to a} f(x)g(x)=\lim_{x\to a}\frac{g(x)}{1/f(x)}$ to convert it to an $\frac{\infty}{\infty}$ form.

*Examples*

4.2.6. Consider $0\times\infty$ form $\lim_{x\to\pi^+}(\pi-x)\tan\frac{x}{2}$.

$$\lim_{x\to\pi^+}(\pi-x)\tan\frac{x}{2}=\lim_{x\to\pi^+}\frac{\pi-x}{\cot\left(\frac{x}{2}\right)}=\lim_{x\to\pi^+}\frac{-1}{-\frac{1}{2}\csc^2\left(\frac{x}{2}\right)}=\lim_{x\to\pi^+}2\sin^2\left(\frac{x}{2}\right)=2$$

4.2.7. Consider $0\times\infty$ form $\lim_{x\to 0^+} x^k\ln x$, where $k > 0$.

$$\lim_{x\to 0^+} x^k\ln x=\lim_{x\to 0^+}\frac{\ln x}{x^{-k}}=\lim_{x\to 0^+}\frac{1/x}{-kx^{-k-1}}=\lim_{x\to 0^+}\frac{1}{-kx^{-k}}=-\frac{1}{k}\lim_{x\to 0^+}x^k=0.$$

***3. Difference***

The limit $\lim_{x\to a}(f(x)-g(x))$, where $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$, is, symbolically, an $\infty-\infty$ *form*. To find the limit, we may first convert the difference into a quotient, then use L'Hopital's rule.

*Examples*

4.2.8. $\lim_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$.

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}=\lim_{x\to 0}\frac{\sin x}{\cos x+\cos x-x\sin x}=0\,.$$

4.2.9. $\lim_{x\to(\pi/2)^+}(\pi\tan x-2x\sec x)$.

$$\lim_{x\to(\pi/2)^+}(\pi\tan x-2x\sec x)=\lim_{x\to(\pi/2)^+}\left(\frac{\pi\sin x}{\cos x}-\frac{2x}{\cos x}\right)=\lim_{x\to(\pi/2)^+}\frac{\pi\sin x-2x}{\cos x}=\lim_{x\to(\pi/2)^+}\frac{\pi\cos x-2}{-\sin x}=2.$$

***4. Power***

The limit $\lim_{x\to a} f(x)^{g(x)}$, where $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$ (the $\infty^0$ form), or $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\infty$ (the $1^\infty$ form). To find the limit, we take the logarithm of the function and find the limit $\lim_{x\to a}\ln f(x)^{g(x)}=\lim_{x\to a} g(x)\ln(f(x))=L$ (if it exists) by L'Hopital's rule, then $\lim_{x\to a} f(x)^{g(x)}=e^L$.

*Examples*

4.2.10. $\lim_{x\to\infty}\left(1+\dfrac{a}{x}\right)^x$.

This is a $1^\infty$ form. Take the logarithm: $\ln\left(1+\dfrac{a}{x}\right)^x=x\ln\left(1+\dfrac{a}{x}\right)$. Use the L'Hopital's rule:

$$\lim_{x\to\infty}x\ln\left(1+\frac{a}{x}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{1/x}=\lim_{x\to\infty}\frac{\left(-\frac{a}{x^2}\right)\Big/\left(1+\frac{a}{x}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{a}{1+\frac{a}{x}}=a\,.$$

Then $\lim_{x\to\infty}\left(1+\dfrac{a}{x}\right)^x=e^a$.

4.2.11. $\lim\limits_{x\to 1^-}(2-x)^{\tan(\pi x/2)}$.

This is a $1^\infty$ form. $\ln(2-x)^{\tan(\pi x/2)} = \tan\dfrac{\pi x}{2}\ln(2-x)$.

$$\lim_{x\to 1^-}\ln(2-x)^{\tan(\pi x/2)} = \lim_{x\to 1^-}\tan\left(\frac{\pi x}{2}\right)\ln(2-x) = \lim_{x\to 1^-}\frac{\ln(2-x)}{\cot\left(\frac{\pi x}{2}\right)} = \lim_{x\to 1^-}\frac{-1/(2-x)}{-\frac{\pi}{2}\sin^{-2}\left(\frac{\pi x}{2}\right)}$$

$$= \frac{2}{\pi}\lim_{x\to 1^-}\frac{\sin^2\left(\frac{\pi x}{2}\right)}{2-x} = \frac{2}{\pi}.$$

Then $\lim\limits_{x\to 1^-}(2-x)^{\tan(\pi x/2)} = e^{2/\pi}$.

4.2.12. $\lim\limits_{x\to\infty}(e^x+x^2)^{1/x}$.

This is a $\infty^0$ form. $\ln(e^x+x^2)^{1/x} = \dfrac{\ln(e^x+x^2)}{x}$.

$$\lim_{x\to\infty}\frac{\ln(e^x+x^2)}{x} = \lim_{x\to\infty}\frac{(e^x+2x)/(e^x+x^2)}{1} = \lim_{x\to\infty}\frac{e^x+2x}{e^x+x^2} = \lim_{x\to\infty}\frac{1+2xe^{-x}}{1+x^2e^{-x}} = 1.$$

Then $\lim\limits_{x\to\infty}(e^x+x^2)^{1/x} = e$.

## § 4.7. OPTIMIZATION PROBLEMS

An optimization problem is to find the global maximum or minimum of a function.

Steps to solve an applied optimization problem:

(i) Identify the quantity to be optimized. If not given, define a symbol to represent this quantity.

(ii) Express this quantity as a function of one or more variables. If not given, define a symbol to represent each of the variables.

(iii) If the number of variables is more than one, use conditions given in the question to eliminate extra variables so that the function, called the *objective function*, is a one-variable function.

(iv) Find the domain of the objective function. Note that, the domain is determined by the question, which is usually NOT the natural domain.

(v) Use calculus to find the global maximum of the objective function.

*Examples*

4.3.1. An open box is made by cutting four square corners of a square cardboard of dimensions 3 by 3 meters. Find the size of the square corners to be cut to maximize the volume of the box.

Let $x$ be the side length of the corner squares. Then the volume of the box is

$V = (3 - 2x)^2x = 9x - 12x^2 + 4x^3$.

The domain of this function is $0 \le x \le \frac{3}{2}$.

Let $V' = 9 - 24x + 12x^2 = 0$, i.e., $3 - 8x + 4x^2 = 0$. Then $x = \frac{8 \pm \sqrt{64-48}}{8} = \frac{8 \pm 4}{8} = \frac{3}{2}, \frac{1}{2}$.

By the first derivative test, $V$ attains a local maximum at $x = \frac{1}{2}$. $V\left(\frac{1}{2}\right) = 2$. Since $V(0) = V\left(\frac{3}{2}\right) = 0$, this local maximum is also a global maximum.

4.3.2. A farmer wants to use fence to enclose a rectangular region of area $800\text{m}^2$ against a wall. Find the dimensions of the region that minimizes the total length of fence.

Let $x$ be the length of the side parallel to the wall and $y$ be the length of the side perpendicular to the wall. Then the total length of fence is $L = x + 2y$. Since the area of the region is $A = xy = 800$, $x = \frac{800}{y}$ and $L = \frac{800}{y} + 2y$. The domain of this function is $0 < y < \infty$.

Let $L' = -\frac{800}{y^2} + 2 = 0$. Then $y^2 = 400$, $y = 20$, and $x = 40$. By the first derivative test, we see that function $L$ attains a local minimum at $x = 40$. Since $L$ approaches 0 when $y$ approaches 0 or infinity, this local minimum is also a global minimum.

4.3.3. A company wants to make a can of the shape of a cylinder. The material to make the top and the bottom costs 0.004 cent per $\text{cm}^2$, and the material to make the lateral surface costs 0.03 cent per $\text{cm}^2$. The total cost the company to spend for each can is $6\pi$ cents. Find the radius and the height of the can that maximize the capacity.

Let $R$ be the radius and let $H$ be the height of the can. Then the capacity of the can is $V = \pi R^2H$.

The area of the top and the bottom is $A_1 = 2\pi R^2$, and the area of the lateral surface is $A_2 = 2\pi RH$. The total cost of the can is $C = 0.04A_1 + 0.03A_2 = 0.08\pi R^2 + 0.06\pi RH = 6\pi$. Hence, $0.08r^2 + 0.06RH = 6$. Solve this equation for $H$.

$$H = \frac{6 - 0.08R^2}{0.06R}.$$

Then $V = \dfrac{6\pi - 0.08\pi R^2}{0.06} = 100\pi R - \dfrac{4}{3}\pi R^3$.

Since $R > 0$, and $H = \dfrac{6 - 0.08R^2}{0.06R} > 0$, i.e., $0.08R^2 < 6$, or $R < \sqrt{\dfrac{6}{0.08}} = \sqrt{75} = 5\sqrt{3}$, the domain of the objective function is $0 < R < 5\sqrt{3}$.

Let $V' = 100\pi - 4\pi R^2 = 0$. $R = \sqrt{\dfrac{100}{4}} = \sqrt{25} = 5$ cm. Then $H = \dfrac{6 - 0.08\times 25}{0.06\times 5} = \dfrac{4}{0.3} = \dfrac{40}{3} \approx 13.3$ cm.

Since $V' > 0$ when $R < 5$, and $V' < 0$ when $R > 5$, function $V(R)$ attains a local maximum at $R = 5$. Since $\lim_{R\to 0} V(R) = \lim_{R\to 5\sqrt{3}} V(R) = 0$, this local maximum is a global maximum.

4.3.4. Find the point on the curve $y^2 = 2x$ that is closest to the point (1, 4).

Let this point be $P = (x, y)$.

The distance between $P$ and (1, 4) is

$$d = \sqrt{(x-1)^2 + (y-4)^2}.$$

Since $x = y^2 / 2$,

$$d = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}.$$

To simplify out calculation, instead of finding the minimum value of $d$, we find the optimal value of

$$D = d^2 = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2 = \frac{y^4}{4} - y^2 + 1 + y^2 - 8y + 16 = \frac{y^4}{4} - 8y + 17.$$

The domain of this function $D(y)$ is $-\infty < y < \infty$.

Let $D' = y^3 - 8 = 0$. Then $y = 2$, and $x = \dfrac{y^2}{2} = 2$. At this point, $D(2, 2) = 5$.

Since $D' < 0$ when $y < 2$, and $D' > 0$ when $y > 2$, function $D(y)$ attains a local minimum at $y = 2$.

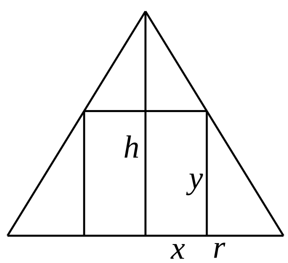
Since $\lim_{y\to-\infty} D(y) = \lim_{y\to\infty} D(y) = \infty$, this local minimum is a global minimum.

The shortest distance from (2, 2) to the parabola $y^2 - 2x = 0$ is $d = \sqrt{5}$.

4.3.5. A right circular cylinder is inscribed in a cone with height $h$ and base radius $r$. Find the maximum volume of the cylinder.

Let $x$ be the radius of the cylinder, and let $y$ be the height of the cylinder. Let $V$ be the volume of the cylinder.

A vertical cross section from the top of the cone is shown in the following figure:




The volume of the cylinder is $V = \pi x^2 y$. By similar triangles, $\dfrac{y}{r-x} = \dfrac{h}{r}$. Hence, $y = \dfrac{h}{r}(r-x)$, and $V = \pi h x^2 \dfrac{r-x}{r} = \dfrac{\pi h}{r}(rx^2 - x^3)$. The domain of this function $V(x)$ is $0 \le x \le r$.

Let $V' = \dfrac{\pi h}{r}(2rx - 3x^2) = 0$. Then $x = \dfrac{2}{3}r$, $y = \dfrac{1}{3}h$, and $V = \dfrac{4}{27}\pi r^2 h$.

Since $V' > 0$ when $x < \dfrac{2}{3}r$, and $V' < 0$ when $x > \dfrac{2}{3}r$, $V(x)$ attains a local maximum at $x = \dfrac{2}{3}r$.

Since $V(0) = V(r) = 0$, this local maximum is also a global maximum.

4.3.6. A rectangular poster is of area 6912 cm$^2$. Side margins are 8 cm and the top and bottom margins are 6 cm. Find the dimensions of the poster that give the maximum printing area.

*Solution*. Let $h$ be the height and let $w$ be the width of the poster. Then the printing area of the poster is $A = (h - 12)(w - 16)$.

Since $wh = 6912$, $A = hw - 16h - 12w + 192 = 6912 - 16h - 12w + 192 = 7104 - \dfrac{16 \times 6912}{w} -$ $12w$, $w \geq 16$. Since $h = \dfrac{6912}{w} \geq 12$, $w \leq 576$. Therefore, the objective function is

$A = 7104 - \dfrac{16 \times 6912}{w} - 12w$, $16 \leq w \leq 576$.

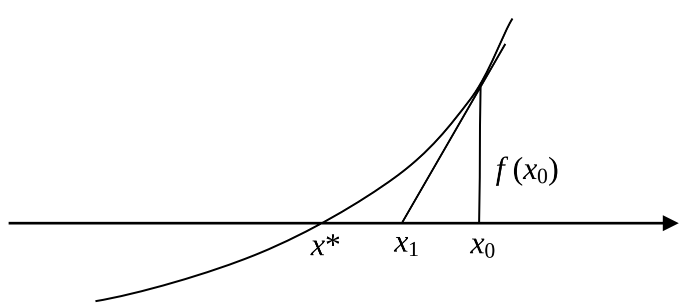
Let $A' = \dfrac{16 \times 6912}{w^2} - 12 = 0$. We have $w^2 = \dfrac{16 \times 6912}{12} = 9216$. Then $w = 96$, $h = 72$, and $A =$ 4800 cm$^2$. Since $A' > 0$ when $w < 96$, and $A' < 0$ when $w > 96$, function $A(w)$ attains a local maximum at $w = 96$. Since $A(16) = A(576) = 0$ this local maximum is a global maximum.

## § 4.8. NEWTON'S METHOD

Newton's method is used to find a root of an equation $f(x) = 0$, using an initial approximation of the root.

By the linearization of the function $y = f(x)$ near a point $x_0$, when $x$ is close to $x_0$, $f(x) \approx f'(x_0)(x - x_0) + f(x_0)$. Suppose $x^*$ is a root of the equation $f(x) = 0$, i.e., $f(x^*) = 0$, and $x_0$ is an approximation of $x^*$. Substituting $x^*$ for $x$ in the linearization, $f'(x_0)(x^* - x_0) + f(x_0) \approx 0$. Hence, $x^* \approx x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$. Then $x_1$ is a better approximation of $x^*$.

This formula can also be illustrated by the following figure:




Since $f'(x_0)$ is the slope of the tangent line at $x_0$, we see that the distance between $x_0$ and $x_1$ is $\dfrac{f(x_0)}{f'(x_0)}$. Repeatedly using this method, we can find an accurate approximation of a root of this equation. The procedure can stop when (a) a number of iteration has been performed, or (b) $f(x_n)$ is small enough, or (c) $|x_n - x_{n-1}|$ is small enough.

*Examples*

4.4.1. Find an approximation of $\sqrt{2}$ accurate up to the second place after the decimal point.

This question is equivalent to find a root of the equation $x^2 = 2$. Then $f(x) = x^2 - 2$, and $f'(x) = 2x$.

The iteration formula is $x_{n+1} = x_n - \dfrac{x_n{}^2 - 2}{2x_n}$.

Let $x_0 = 2$. Then

$$x_1 = 2 - \frac{2^2 - 2}{2 \times 2} = 1.5;$$

$$x_2 = 1.5 - \frac{1.5^2 - 2}{2 \times 1.5} \approx 1.41667;$$

$$x_3 = 1.41667 - \frac{1.41667^2 - 2}{2 \times 1.41667} \approx 1.41422.$$

Since the second place after the decimal point in $x_2$ and $x_3$ are the same. Stop at $x_3$.

4.4.2. Find a root of the equation $4\ln x = x + 1$, with $x_0 = 2$ by Euler's method. Stop when the difference between two consecutive approximations is less than 0.001.

We have $f(x) = 4\ln x - x - 1$, and $f'(x) = \dfrac{4}{x} - 1$. The iteration formula is
$x_{n+1} = x_n - \dfrac{4\ln x_n - x_n - 1}{4/x_n - 1}$.

$x_0 = 2$.

$$x_1 = 2 - \frac{4\ln 2 - 2 - 1}{0.25 \times 2 - 1} \approx 2.2274;$$

$$x_2 = 2.2741 - \frac{4\ln 2.2741 - 2.2741 - 1}{0.25 \times 2.2741 - 1} \approx 2.2576;$$

$$x_3 = 2.5763 - \frac{4\ln 2.5763 - 2.5763 - 1}{0.25 \times 2.5763 - 1} \approx 2.2581.$$

Since $|x_3 - x_2| \approx 0.0005 < 0.001$. Let $x_3$ be the approximation.

4.4.3. Use Euler's method to find an approximation of a root of the equation $x^3 - x + 1 = 0$ starting with $x_0 = -1$. Stop when the value of $f(x) = x^3 - x + 1 < 0.0001$.

The iteration formula is $x_{n+1} = x_n - \dfrac{x_n{}^3 - x_n + 1}{3x_n{}^2 - 1}$.

$x_0 = -1$, $f(x_0) = 1$;

$x_1 = 1 - \dfrac{(-1)^3 - (-1) + 1}{3(-1)^2 - 1} \approx -1.5$, $f(x_1) = -0.875$;

$x_2 = -1.5 - \dfrac{(-1.5)^3 - (-1.5) + 1}{3(-1.5)^2 - 1} \approx -1.3478$, $f(x_2) = -0.1001$;

$x_3 = -1.3478 - \dfrac{(-1.3478)^3 - (-1.3478) + 1}{3(-1.3478)^2 - 1} \approx -1.3252$, $f(x_3) = -0.00206$;

$x_4 = -1.3252 - \dfrac{(-1.3252)^3 - (-1.3252) + 1}{3(-1.3252)^2 - 1} \approx -1.3247$, $f(x_4) = -0.00008$.

We have a very accurate root $x = 1.3247$ of this equation.