

LECTURE NOTES

Based on Calculus, Early Transcendentals, by James Stewart

MAT 1320 Calculus I

Fall 2019

CHAPTER 1. FUNCTIONS AND MODELS

§ 1.1. FOUR WAYS TO REPRESENT A FUNCTION

1. Definition of Functions

A *function* is defined by three parts: A *domain* D , which is a subset of real numbers usually consisting of one or more intervals, a *range* R , which is also a subset of real numbers usually consisting of one or more intervals, and a relation f that assigns a **unique** number in R to **each** number in D , and every number in R is assigned to at least one number in D . If a function f has domain D and range R , we also say f is a function *from* D *to* R .

If we use x to represent an arbitrary value in D , and use y to denote the value in R that is assigned to x , then we write $y = f(x)$, where x is called the *independent variable*, and y is called the *dependent variable*.

If $y = f(x)$, we call y the *image* of x , and x a *pre-image* of y . Then, according to the definition of the function, every x in D has a unique image in R , but a value y in R may have one or more pre-images in D .

In many cases, a function is defined by an expression or a number of expressions. The *natural domain* of a function is the set of all values of x such that the expression(s) is defined. If the domain is not otherwise specified, the "domain" of a function means the "natural domain".

Examples

1.1.1. $h = 65 + 12t - 5t^2, 0 \leq t \leq 6.$

The independent variable is t , and the dependent variable is h . The domain of this function is specified to be the set of real numbers in interval $[0, 6]$, although the natural domain of this function is the set of all real numbers.

1.1.2. $y = \frac{\sqrt{4-x^2}}{x^2+x}.$

Since we must have $x^2 + x \neq 0$, and $4 - x^2 \geq 0$, the (natural) domain of this function consists of three intervals $[-2, -1)$, $(-1, 0)$, and $(0, 2]$.

2. Graphs of Equations and Functions

Let $F(x, y) = G(x, y)$ be an equation of two variables x and y . A pair of values (x_0, y_0) is a *solution* to this equation if $F(x_0, y_0) = G(x_0, y_0)$. With the Cartesian coordinate system on a plane, every solution (x_0, y_0) of this equation corresponds to a point on the plane. The points of all solutions to this equation usually form a *curve*, called the *graph* of the equation. For example, we know that the graph of the equation $x^2 + y^2 = 1$ is the *unit circle*.

If a function $y = f(x)$ is given, every pair of x and $y = f(x)$ corresponds to a point (x, y) on the plane. All these points usually form a curve, called the *graph* of the function.

A curve is the graph of a function if every vertical line intersects the curve at at most one point. This is called the *vertical line test*.

Intersection of the Graphs of Two Functions

The intersections of the graphs of two functions $y = f(x)$ and $y = g(x)$ can be found by solving the equation $f(x) = g(x)$.

The x coordinates of intersections of the graph of $y = f(x)$ and the x -axis are *x-intercepts*, and the y coordinate of the intersection of the graph of $y = f(x)$ and the y -axis is the *y-intercept*. A function may have any number of x -intercepts, but exactly one y -intercept if $x = 0$ is in the domain of this function, or no y -intercept if $x = 0$ is not in the domain of this function. The x -intercepts are found by solving the equation $f(x) = 0$, and the y -intercept is $y_0 = f(0)$.

3. Properties of Functions

Let $y = f(x)$ be a function defined in an interval (a, b) .

Monotonicity

If, in interval (a, b) , $x_1 > x_2$ implies $f(x_1) > f(x_2)$, i.e., a bigger x corresponds to a bigger y , and a smaller x corresponds to a smaller y , then $f(x)$ is *increasing* in (a, b) . If, in (a, b) , $x_1 > x_2$ implies $f(x_1) < f(x_2)$, i.e., a bigger x corresponds to a smaller y , and a smaller x corresponds to a bigger y , then $f(x)$ is *decreasing* in (a, b) .

Local and Global Extrema

If there exists an interval (a, b) in the domain of a function $y = f(x)$ and $a < x_0 < b$ such that $f(x_0) \geq f(x)$ for every value x in (a, b) , then this function attains a *local maximum* $f(x_0)$ (also called a *relative maximum*) at $x = x_0$; if $f(x_0) \leq f(x)$ for every value x in (a, b) , then this function attains a *local minimum* $f(x_0)$ (also called a *relative minimum*) at $x = x_0$.

If, for every x in the domain of this function $f(x_0) \geq f(x)$, then this function attains a *global maximum* $f(x_0)$ (also called an *absolute maximum*) at $x = x_0$. If, for every x in the domain of this





function $f(x_0) \leq f(x)$, then this function attains an *global minimum* $f(x_0)$ (also called an *absolute minimum*) at $x = x_0$.

Concavity and Inflection Points

If, in an interval (a, b) , this function increases faster and faster, or decreases slower and slower, we say that (the graph of) this function is *concave up* in (a, b) . If, in (a, b) , this function increases slower and slower, or decreases faster and faster, we say that (the graph of) this function is *concave down* in (a, b) . The graph is concave up if it looks like a part of a cup. The graph is concave down if it looks like a part of a cap.

If interval (a, b) is in the domain of a function $f(x)$, and $a < x_0 < b$, such that the graph of $f(x)$ is concave up when $a < x < x_0$ and it is concave down when $x_0 < x < b$, or the graph of $f(x)$ is concave down when $a < x < x_0$ and it is concave up when $x_0 < x < b$, then we say that $f(x)$ has an *inflection point* $(x_0, f(x_0))$ at $x = x_0$.

With different monotonicity and different concavity, the shapes of the graph of a function are summarized in the following table:

	increasing	decreasing
concave up		
concave down		

Even/odd Functions and Their Graphs

If a function $y = f(x)$ satisfies the property $f(-x) = f(x)$, then $f(x)$ is an *even function*; if $f(-x) = -f(x)$, then it is an *odd function*. For example, $y = x^n$ is an even function if n is an even number, and it is an odd function if n is an odd number. Function $y = \sin x$ is an odd function, and function $y = \cos x$ is an even function.

The graph of an even function is symmetric with respect to the y -axis; the graph of an odd function is symmetric with respect to the origin.

§ 1.2. MATHEMATICAL MODELS - A CATALOG OF FUNCTIONS

1. Linear Functions

Linear Functions and Their Graphs

A linear function has the form $y = mx + b$, where m and b are constants.

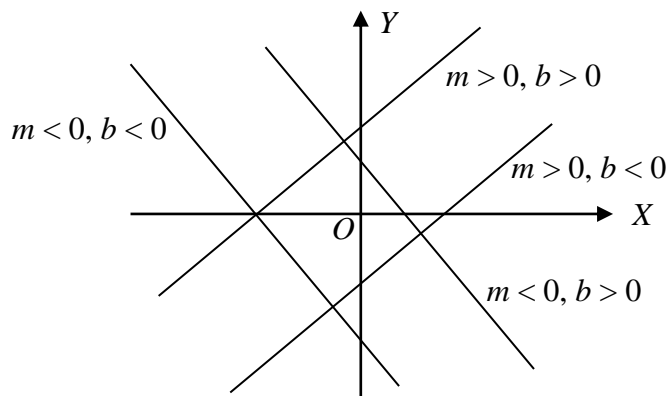
Let x_1 and x_2 be two different values of x . Then $y(x_1) = mx_1 + b$, and $y(x_2) = mx_2 + b$. The *rate of change* of this function $\frac{y(x_2) - y(x_1)}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = m$ is called the *slope* of this linear function. The rate of change of a linear function is a constant, i.e., independent of x .

The x -intercept of $y = mx + b$ is $x_0 = -b / m$, and its y -intercept is b .

The graph of a linear function is a straight line, but not all straight lines are graphs of functions.

The equation of a straight line is a linear equation of the form $ax + by = c$. If $b \neq 0$, this equation can be written as a function $y = -\frac{a}{b}x + \frac{c}{b}$. The slope of this function is $m = -\frac{a}{b}$. If $b = 0$, the equation has the form $ax = b$. The graph of this equation is a vertical line, which is not the graph of a function.

Lines with different signs of the slope and the intercept are illustrated in the following figure:



Two lines with slopes m_1 and m_2 , respectively, are parallel if $m_1 = m_2$.

Two lines with slopes m_1 and m_2 , respectively, are perpendicular if $m_1 m_2 = -1$.

Finding the Equation of a Line

The equation of a line with slope m passing through a point (x_0, y_0) has the form $y = mx + b$. Then $y_0 = mx_0 + b$, $b = y_0 - mx_0$. The equation of the line is $y = mx - mx_0 + y_0$ or $y = m(x - x_0) + y_0$. This is called the *slope-point form* of the equation of a line.

If (x_1, y_1) and (x_2, y_2) are two different points on the graph of a linear function, then $x_1 \neq x_2$, the slope of this line is $m = \frac{y_2 - y_1}{x_2 - x_1}$, and the equation of this line is $y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$. This is called the *two-point form* of the equation of a line.

Examples

1.2.1. Find the equation of a line passing through two points $(-5, 4)$ and $(4, -2)$.

$$\text{The slope } m = \frac{-2 - 4}{4 - (-5)} = -\frac{2}{3}.$$

The equation of the line has the form $y = -\frac{2}{3}x + b$. Use the coordinates of a point:

$$4 = -\frac{2}{3} \times (-5) + b. \quad b = 4 - \frac{10}{3} = \frac{2}{3}. \quad \text{The equation is } y = -\frac{2}{3}x + \frac{2}{3}, \text{ or } 3y = -2x + 2, \text{ or } 2x + 3y = 2.$$

1.2.2. Find the equations of two lines going through a point $(2, -4)$ parallel and perpendicular, respectively, to the line L with equation $x + 2y = 5$.

The slope of L is $m = -\frac{1}{2}$. The equation of the line parallel to this line has the form

$$y = -\frac{1}{2}x + b. \quad \text{Since } -4 = -1 + b, \quad b = -3. \quad \text{The equation of the line parallel to } L \text{ is } y = -\frac{1}{2}x - 3, \text{ or } x + 2y = -6.$$

The slope of the line perpendicular to L is $-\frac{1}{m} = -\frac{1}{-1/2} = 2$. The equation of this line has the form $y = 2x + b$. Then $-4 = 4 + b$. $b = -8$. The equation of the line perpendicular to L is $y = 2x - 8$, or $2x - y = 8$.

2. Quadratic Functions

Quadratic Polynomial, Roots, and Factorization

The general form of a *quadratic polynomial* is $ax^2 + bx + c$, $a \neq 0$.

The roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Let $D = b^2 - 4ac$.

If $D > 0$, this quadratic equation has two real roots $x_1 = \frac{-b + \sqrt{D}}{2a}$, $x_2 = \frac{-b - \sqrt{D}}{2a}$, and $ax^2 + bx + c = a(x - x_1)(x - x_2)$.

If $D = 0$, this quadratic equation has only one root $x_0 = \frac{-b}{2a}$ (we also say two identical roots $x_1 = x_2 = \frac{-b}{2a}$), and $ax^2 + bx + c = a(x - x_0)^2$.

If $D < 0$, this quadratic equation does not have (real) roots. It cannot be factorized into a product of two linear factors with real coefficients. In this case, it is said to be *irreducible*.

Examples

1.2.3. $Q(x) = 3x^2 - 4x - 4$.

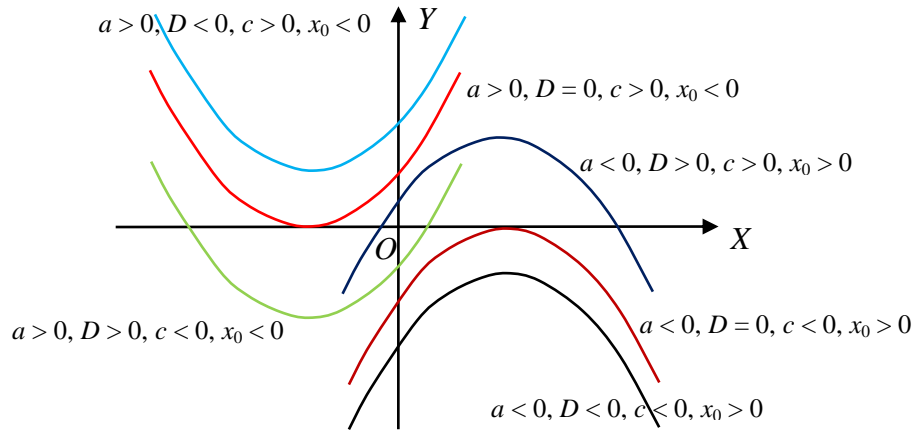
Let $Q(x) = 0$. The roots of this equation are $x = 2, -\frac{2}{3}$.

$$Q(x) = 3(x - 2)\left(x + \frac{2}{3}\right) = (x - 2)(3x + 2).$$

Quadratic Functions and Their Graphs

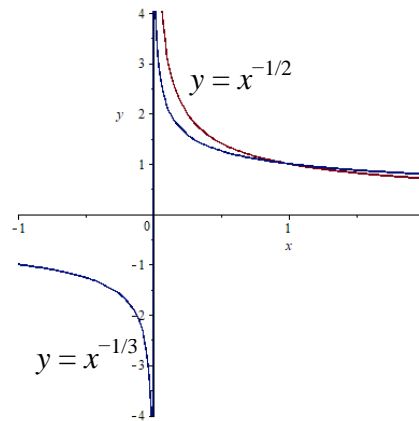
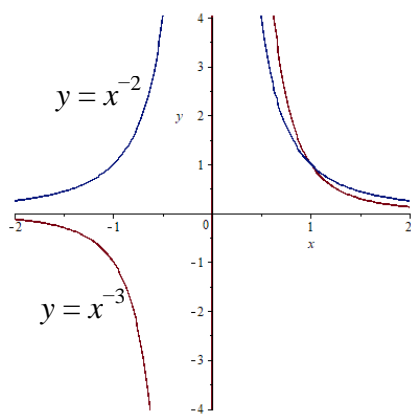
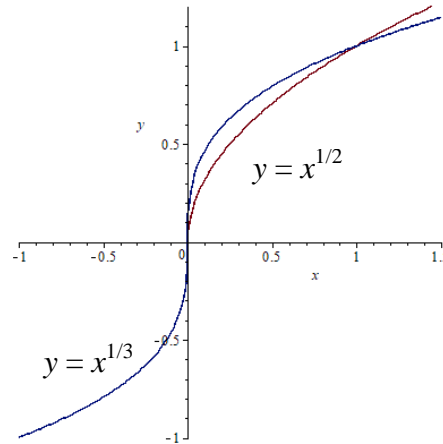
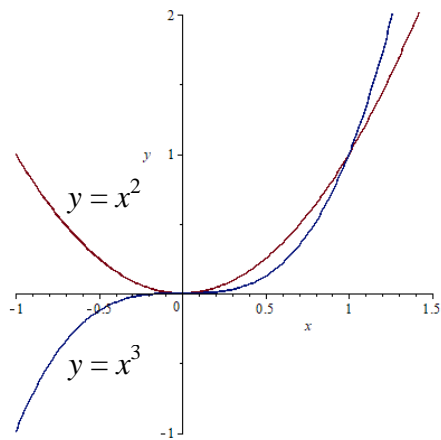
The graph of a quadratic function $y = ax^2 + bx + c$ is a parabola. If $a > 0$, the parabola opens upwards; if $a < 0$, the parabola opens downwards. The *vertex* of the parabola is at (x_0, y_0) , where $x_0 = -\frac{b}{2a}$, $y_0 = \frac{4ac - b^2}{4a}$. The x -intercepts are the roots of the equation $ax^2 + bx + c = 0$, and the y -intercept is c .

The following figure illustrates some typical cases of a parabola:



3. Power Functions

Power functions have the form $y = x^k$, $k \neq 0$. The following are the graphs of some typical cases of power functions:



4. Polynomial Functions

Polynomial Functions

Function $y = c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$, $a_n \neq 0$, is a *polynomial function* of *degree* n . The term $c_n x^n$ is called the *leading term*, and c_n is the *leading coefficient*.

The natural domain of a polynomial function is the set of all real numbers.

The y -intercept of this polynomial function is $y(0) = c_0$. The x -intercepts are the roots of the equation $P(x) = 0$, which are also called the *zeros* of the polynomial $P(x)$. A polynomial of degree n may have at most n x -intercepts.

5. Rational Functions

Rational functions have the form $f(x) = \frac{M(x)}{N(x)}$, where $M(x)$ and $N(x)$ are polynomials. It is a *proper rational function* if the degree of $M(x)$ is less than the degree of $N(x)$. Otherwise, it is an *improper rational function*. If $y = \frac{M(x)}{N(x)}$ is an improper rational function, using long division to divide $M(x)$ by $N(x)$, we have $M(x) = N(x)Q(x) + R(x)$, where $Q(x)$ is the quotient and $R(x)$ is the remainder. Then $\frac{M(x)}{N(x)} = Q(x) + \frac{R(x)}{N(x)}$, where $\frac{R(x)}{N(x)}$ is a proper rational function.

The natural domain of a rational function $f(x) = \frac{M(x)}{N(x)}$ is the set of all real numbers except the zeros of the polynomial $N(x)$. If $M(x_0) = 0$ but $N(x_0) \neq 0$, then $f(x_0) = 0$, and x_0 is an x -intercept.

Remainder Theorem: $P(a)$ is the remainder of dividing $P(x)$ by $x - a$.

In other words, $P(x) = (x - a)Q(x) + P(a)$, where $Q(x)$ is the quotient of $P(x)$ divided by $(x - a)$.

Corollary: If $P(a) = 0$, then $x - a$ is a factor of $P(x)$.

6. Algebraic Functions

A function that involves arithmetic operations and/or roots is an *algebraic function*. Polynomials and rational functions are algebraic functions. As examples of algebraic functions involving

radicals, $y = \sqrt{x^2 + 1}$ and $y = \frac{x^{1/3}}{x^{2/3} + 1}$ are algebraic functions.

7. Transcendental Functions

A function that is not an algebraic function is a *transcendental function*. Trigonometric functions, exponential functions, and logarithmic functions are transcendental functions.

§ 1.3. NEW FUNCTIONS FROM OLD FUNCTIONS

1. Transformations

Let $y = f(x)$ be a function.

Shift Operations

$y = f(x + a)$ is a new function called a *horizontal shift* of $f(x)$. The graph of a horizontal shift can be obtained by shifting the graph of $f(x)$ to the left by a units if $a > 0$, or to the right by $|a|$ units if $a < 0$. More precisely, if (x_0, y_0) is a point on the graph of $y = f(x)$, then $(x_0 - a, y_0)$ is a point on the graph of $y = f(x + a)$.

$y = f(x) + b$ is a new function called a *vertical shift* of $f(x)$. The graph of a vertical shift can be obtained by shifting the graph of $f(x)$ up by b units if $b > 0$, or down by $|b|$ units if $b < 0$. More precisely, if (x_0, y_0) is a point on the graph of $y = f(x)$, then $(x_0, y_0 + b)$ is a point on the graph of $y = f(x) + b$.

Stretch / Shrink Operations

Let $k > 1$.

$y = f\left(\frac{x}{k}\right)$ is a new function called a *horizontal stretch* of $f(x)$ by a factor k . The graph of a horizontal stretch can be obtained by stretching the graph of $f(x)$ horizontally by a factor k . More precisely, if (x_0, y_0) is a point on the graph of $y = f(x)$, then (kx_0, y_0) is a point on the graph of $y = f\left(\frac{x}{k}\right)$.

$y = f(kx)$ is a new function called a *horizontal shrink* of $f(x)$ by a factor k . The graph of a horizontal shrink can be obtained by compressing the graph of $f(x)$ horizontally by a factor k . More precisely, if (x_0, y_0) is a point on the graph of $y = f(x)$, then $\left(\frac{x_0}{k}, y_0\right)$ is a point on the graph of $y = f(kx)$.

$y = kf(x)$ is a new function called a *vertical stretch* of $f(x)$ by a factor k . The graph of a vertical stretch can be obtained by stretching the graph of $f(x)$ vertically by a factor k . More precisely, if (x_0, y_0) is a point on the graph of $y = f(x)$, then (x_0, ky_0) is a point on the graph of $y = kf(x)$.

$y = \frac{1}{k}f(x)$ is a new function called a *vertical shrink* of $f(x)$ by a factor k . The graph of a vertical shrink can be obtained by compressing the graph of $f(x)$ vertically by a factor k . More precisely, if (x_0, y_0) is a point on the graph of $y = f(x)$, then $\left(x_0, \frac{y_0}{k}\right)$ is a point on the graph of $y = f(kx)$.

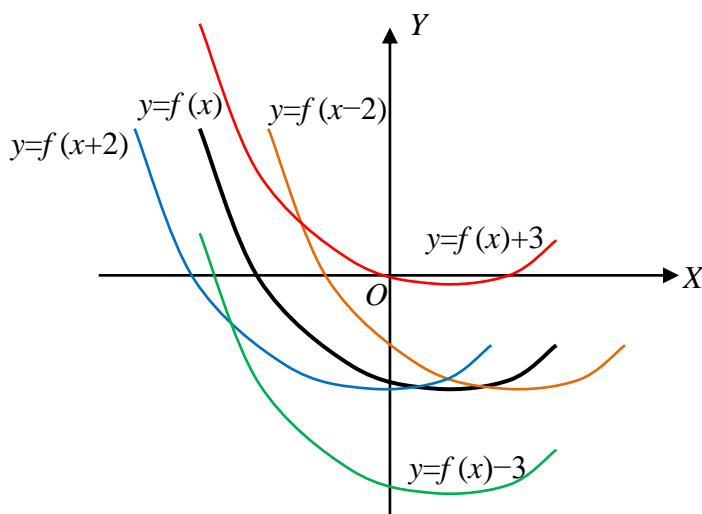
Reflection Operations

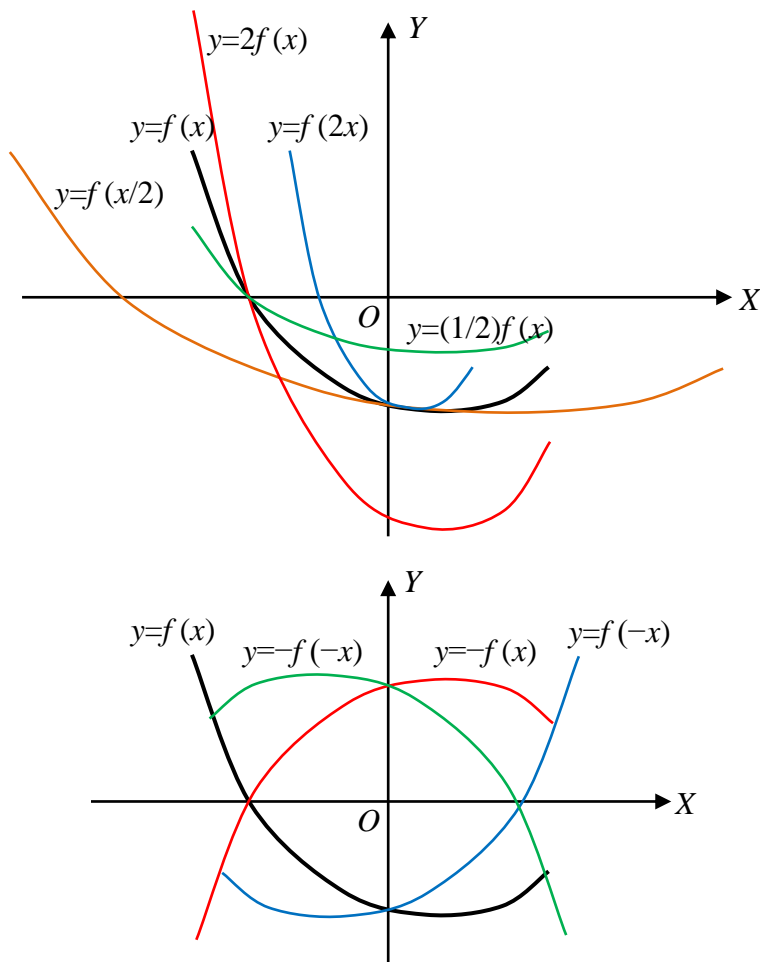
$y = f(-x)$ is a new function called the *horizontal reflection* of $f(x)$. The graph of a horizontal reflection can be obtained by reflecting the graph of $y = f(x)$ about the y -axis. More precisely, if (x_0, y_0) is a point on the graph of $y = f(x)$, then $(-x_0, y_0)$ is a point on the graph of $y = f(-x)$.

$y = -f(x)$ is a new function called the *vertical reflection* of $f(x)$. The graph of a vertical reflection can be obtained by reflecting the graph of $y = f(x)$ about the x -axis. More precisely, if (x_0, y_0) is a point on the graph of $y = f(x)$, then $(x_0, -y_0)$ is a point on the graph of $y = -f(x)$.

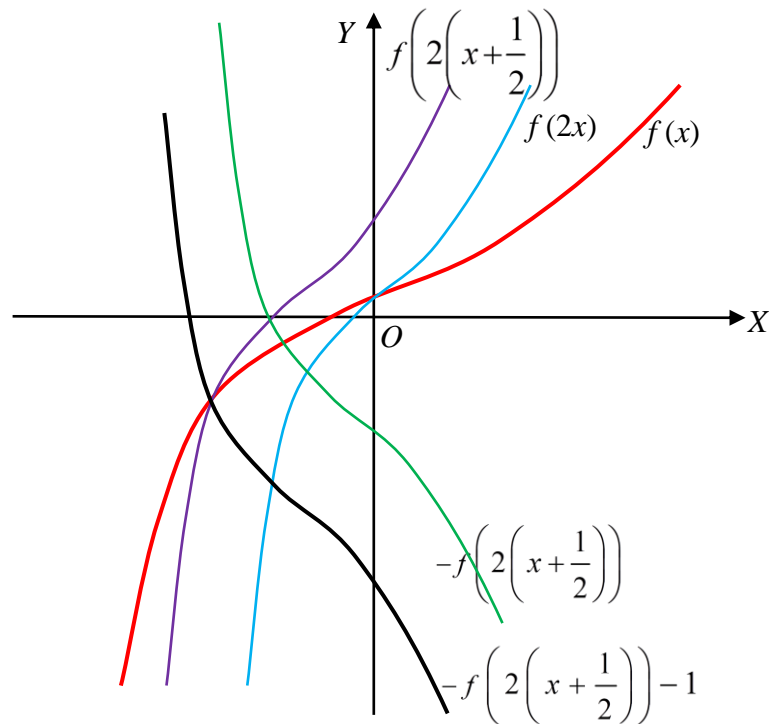
$y = -f(-x)$ is a new function called the *reflection of $f(x)$ about the origin*. The graph of a reflection about the origin can be obtained by rotating the graph of $y = f(x)$ about the origin for an angle π radians (180 degrees). More precisely, if (x_0, y_0) is a point on the graph of $y = f(x)$, then $(-x_0, -y_0)$ is a point on the graph of $y = -f(-x)$.

The following figures illustrate the transformations:





Example 1.3.1. Suppose the graph of a function $y = f(x)$ is given, we can find the graph of the function $y = -f(2x + 1) - 1$ by a sequence of transformation operations. Since $-f(2x + 1) - 1 = -f\left(2\left(x + \frac{1}{2}\right)\right) - 1$. We use the sequence $f(x) \rightarrow f(2x) \rightarrow f\left(2\left(x + \frac{1}{2}\right)\right) \rightarrow -f\left(2\left(x + \frac{1}{2}\right)\right) \rightarrow -f\left(2\left(x + \frac{1}{2}\right)\right) - 1$.



2. Function Arithmetic

Let $f(x)$ and $g(x)$ be two functions. Then

$$(f + g)(x) = f(x) + g(x),$$

$$(f + g)(x) = f(x) + g(x),$$

$$(f - g)(x) = f(x) - g(x),$$

$$(fg)(x) = (f(x))(g(x)),$$

$$(f/g)(x) = f(x) / g(x), g(x) \neq 0.$$

3. Composition

Let $g(x)$ be a function from A to B , and let $f(x)$ be a function from B to C . The function $f(g(x))$ is a function from A to C , called the *composition* of $f(x)$ and $g(x)$, denoted by $(f \circ g)(x)$. To make the composition $(f \circ g)(x)$ explicit, we may use an *intermediate variable* u , and write $y = f(u)$ and $u = g(x)$. Then $y = f(u) = f(g(x)) = (f \circ g)(x)$.

Examples

$$1.3.2. f(x) = x^2. g(x) = x + 1.$$

$$(f \circ g)(x) = f(g(x)) = (x + 1)^2, (g \circ f)(x) = g(f(x)) = x^2 + 1.$$

$$(f \circ f)(x) = f(f(x)) = x^4, (g \circ g)(x) = g(g(x)) = x + 2.$$

$$1.3.3. f(x) = \sqrt{x+1}. g(x) = x^3.$$

$$(f \circ g)(x) = f(g(x)) = \sqrt{x^3+1}, (g \circ f)(x) = g(f(x)) = (x+1)^{3/2}.$$

$$(f \circ f)(x) = f(f(x)) = \sqrt{\sqrt{x+1}+1}, (g \circ g)(x) = g(g(x)) = (x^3)^3 = x^9.$$

1.3.4. The following table gives some values of functions $y = f(x)$, $y = g(x)$, and their compositions:

x	1	2	3	4
$f(x)$	2	3	4	1
$g(x)$	3	1	2	4
$(f \circ g)(x)$	4	2	3	1
$(g \circ f)(x)$	1	2	4	3
$(f \circ f)(x)$	3	4	1	2
$(g \circ g)(x)$	2	3	1	4

Decompose a Function into a Composition of Two Functions

A complex function may be decomposed into the composition of two functions. This is done, in particular, when the last operation of finding y from a given x is not any one of the arithmetic operations.

Examples

$$1.3.5. y = (3x + 2)^{10}. \text{ Let } u = 3x + 2. \text{ Then } y = u^{10}.$$

$$1.3.6. y = \sqrt{x^2 + 1}. \text{ Let } u = x^2 + 1. \text{ Then } y = \sqrt{u}.$$

Although we may have different ways to write a function as a composition of two functions, we will always leave out the "last operation" to define the intermediate variable u so that y is obtained by u with only one operation.

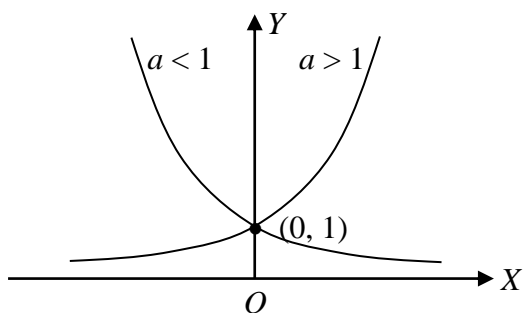
§ 1.4. EXPONENTIAL FUNCTIONS

1. Definition and Graph

Exponential functions have the general form $y = Ca^{kx}$, where C , k , and a are constants, $C \neq 0$, $k \neq 0$, $a > 0$, and $a \neq 1$. The number a is call the *base* of this function.

The graph of the general exponential function $y = Ca^{kx}$ can be obtained from the graph of the function $y = a^x$ by vertical/horizontal stretch / shrink and / or reflection.

The graph of $y = a^x$ is illustrated in the following figure:



2. Properties

From the graph, we can "read" a number of important properties of function $y = a^x$:

- (a) Domain: $-\infty < x < \infty$.
- (b) Range: $y > 0$.
- (c) Monotonicity: When $a > 1$, the function is increasing; when $a < 1$, this function is decreasing.
- (d) Concavity: Always concave up.
- (e) $f(0) = 1$.

3. The Natural Exponential Function

If an irrational number $e = 2.71828\dots$ is used as the base of the exponential function, then this exponential function is called the *natural exponential function*.

APP. D. TRIGONOMETRIC FUNCTIONS

1. Definitions

An angle is put in the Cartesian plane such that the vertex is at the origin, and one side, called the *initial side*, is in the direction of the x -axis. This is called the *standard position* of the angle. The other side, called the *terminal side*, is in the direction of a radius of a circle C centered at the origin.

We may assume that the terminal side is obtained by rotating the initial side counterclockwise around the origin. When the terminal side rotates round the origin, the intersection of the terminal side and circle C rotates around the origin along an arc of the circle. We use the ratio of the length of this arc and the radius of C to be the measurement of the angle, called the *radians*. Obviously, the measurement is independent of the radius of circle C .

The terminal side may also be obtained by rotate the initial side clockwise around the origin, and it is measured by the ratio of the length of a different arc and the radius of C . The measurement of this angle is the negation of the ratio.

Since the terminal side may rotate around the origin by a number of full revolution to reach the final position, if x is a measurement of an angle, then $x + 2n\pi$, where n is an integer, are also measurements of this angle.

In mathematics, angles are always measured by radians.

Let A be an angle in standard position. Let the intersection of the terminal side and the unit circle be (x_0, y_0) . The *trigonometric functions* are defined as follows:

$$\sin A = y_0; \cos A = x_0; \tan A = \frac{y_0}{x_0}; \cot A = \frac{x_0}{y_0}; \sec A = \frac{1}{x_0}; \csc A = \frac{1}{y_0}.$$

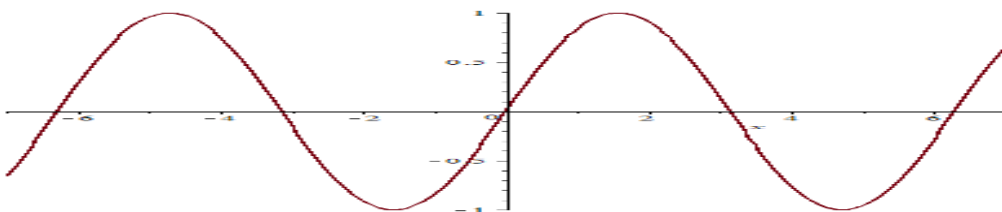
In right triangle, let the length of the hypotenuse be c . the length of the adjacent side of an angle A be b , and let the length of the opposite side of angle A be a . Then

$$\sin A = \frac{a}{c}; \cos A = \frac{b}{c}; \tan A = \frac{a}{b}; \cot A = \frac{b}{a}; \sec A = \frac{c}{b}; \csc A = \frac{c}{a}.$$

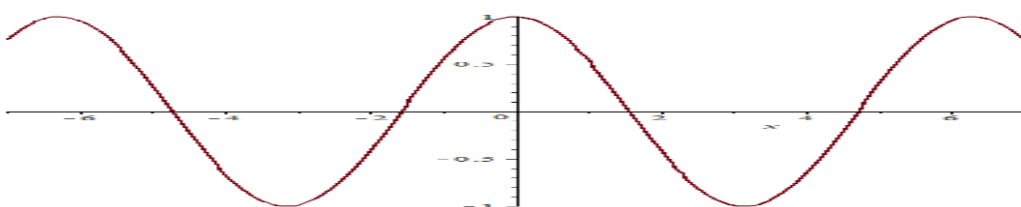
2. Graphs and Properties

The following are graphs of trigonometric functions:

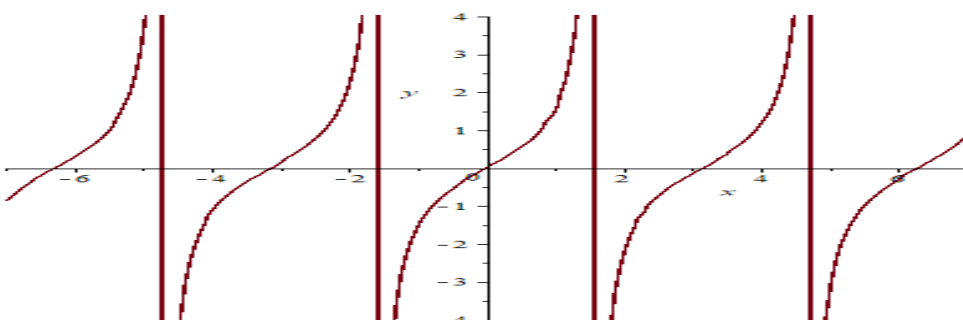
$$y = \sin x$$



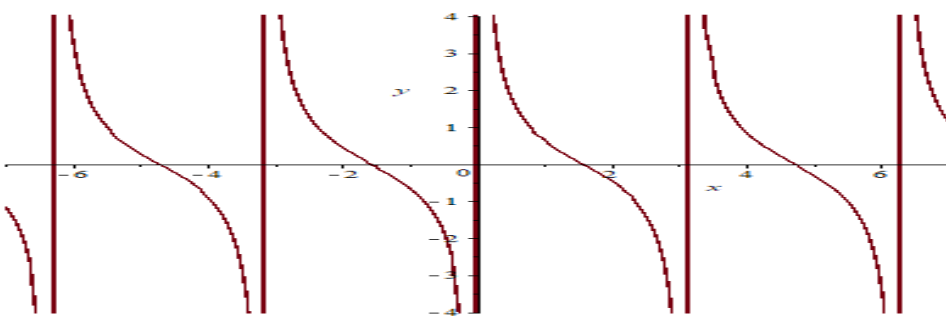
$y = \cos x$



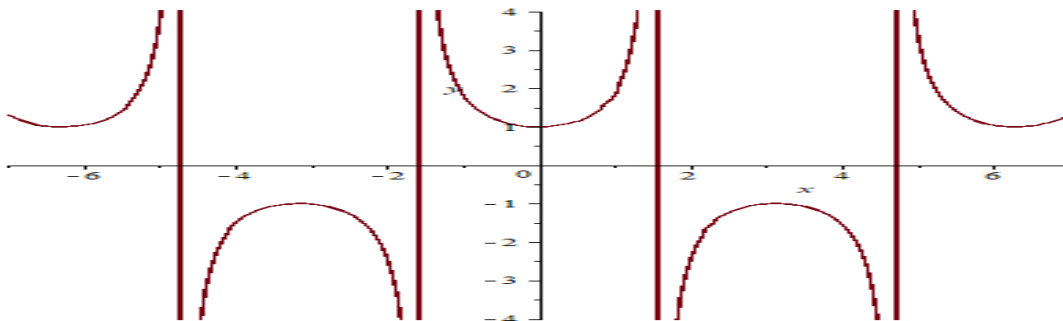
$y = \tan x$



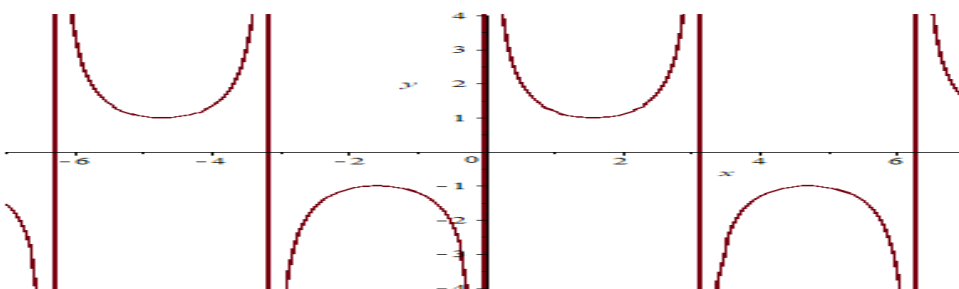
$y = \cot x$



$y = \sec x$



$$y = \csc x$$



Note that $\sin x$, $\tan x$, $\cot x$, and $\csc x$ are odd functions, and $\cos x$ and $\sec x$ are even functions.

Trigonometric functions are periodic. Functions $\sin x$, $\cos x$, $\sec x$, and $\csc x$ have (minimum) period 2π , and $\tan x$ and $\cot x$ have (minimum) period π .

3. Identities

The relations among the trigonometric functions are given by a number of identities. The following are some most important ones:

$$\sin^2 x + \cos^2 x = 1,$$

$$\tan x = \sin x / \cos x = 1 / \cot x,$$

$$\cot x = \cos x / \sin x = 1 / \tan x,$$

$$\sec x = 1 / \cos x,$$

$$\csc x = 1 / \sin x,$$

$$1 + \tan^2 x = \sec^2 x,$$

$$1 + \cot^2 x = \csc^2 x,$$

$$\sin(2x) = 2 \sin x \cos x,$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x,$$

$$\sin^2 x = \frac{1}{2} (1 - \cos(2x)),$$

$$\cos^2 x = \frac{1}{2} (1 + \cos(2x)).$$

§ 1.5. INVERSE FUNCTIONS AND LOGARITHMS

1. The Inverse of a Function

Let $y = f(x)$ be a function from A to B . If every y in the B has exactly one pre-image x in A , this function is said to be a *one-to-one* (1-1) *function*. The graph of a function is the graph of a one-to-one function if and only if every horizontal line intersects the graph at at most one point. This is called the *horizontal line test*.

If $y = f(x)$ is a one-to-one function from A to B , we can find the unique x from every given y in B . This defines a new function from B to A . This is called the *inverse* of f , denoted by f^{-1} .

By definition, the domain of f^{-1} is the range of f , and the range of f^{-1} is the domain of f . We also have the relations $f^{-1}(f(x)) = x$ for all x in the domain of f and $f(f^{-1}(y)) = y$ for every y in the range of f .

If f and g are one-to-one functions, then the composition $f \circ g$ is a one-to-one function, and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

To find the inverse of a function, solve this function as an equation for x , and interchange x and y .

Examples

1.6.1. $y = 3x + 1$.

$$x = \frac{1}{3}(y - 1). \text{ The inverse is } y = \frac{1}{3}(x - 1).$$

1.6.2. $y = \frac{2x-1}{x+3}$.

$$xy + 3y = 2x - 1, (y - 2)x = -3y - 1, x = -\frac{3y+1}{y-2}. \text{ The inverse is } y = -\frac{3x+1}{x-2}.$$

$$1.6.3. y = \sqrt{x^2 + 1}.$$

$x^2 = y^2 - 1$, $x = \pm \sqrt{y^2 - 1}$. This function does not have an inverse.

If (a, b) is a point on the graph of $y = f(x)$, i.e., $b = f(a)$, since $a = f^{-1}(b)$, (b, a) is a point on the graph of $y = f^{-1}(x)$. Hence, the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$.

2. The Logarithm

If $a^b = c$, where $a > 0$, and $a \neq 1$, then b is the *logarithm* of c with *base* a , denoted by $b = \log_a c$. Since $c = a^b > 0$, only positive numbers have the logarithm.

If e is used as the base, the logarithmic is called the *natural logarithm*. We use symbol $\ln x$ to denote $\log_e x$.

Properties of Logarithm

(a) $\log_a 1 = 0$. (In particular, $\ln 1 = 0$).

(b) $\log_a a = 1$. (In particular, $\ln e = 1$).

Assume $x > 0$, $y > 0$.

(c) $\log_a (xy) = \log_a x + \log_a y$.

(d) $\log_a (x/y) = \log_a x - \log_a y$.

(e) $\log_a x^k = k \log_a x$.

(f) $a^{\log_a x} = x$. (In particular, $e^{\ln x} = x$).

(g) $\log_a x = \frac{\log_b x}{\log_b a}$. (The base changing formula)

In particular, when $b = e$, $\log_a x = \frac{\ln x}{\ln a}$.

With the properties of the logarithm, we can solve equations with exponents and/or logarithm.

Examples

1.6.5. Solve $\log_2(x + 1) - \log_2(2x - 3) = 2$.

$$\log_2 \frac{x+1}{2x-3} = 2, \frac{x+1}{2x-3} = 4, 8x - 12 = x + 1, 7x = 13, x = \frac{13}{7}. \text{ Since } x + 1 = \frac{13}{7} + 1 > 0, \text{ and } 2 \times \frac{13}{7} - 3 = \frac{5}{7} > 0. x = \frac{13}{7} \text{ is a root of this equation.}$$

$$1.6.6. \log_2 x + \log_2(x + 1) = 1.$$

$\log_2 x(x + 1) = 1, x(x + 1) = 2, x^2 + x - 2 = 0.$ Solving this equation, we have $x = 1$, and $x = -2$. We see that $x = 1$ is a root of this equation. However, since, when $x = -2$, $\log_2 x$ and $\log_2(x + 1)$ are not defined, $x = -2$ is not a root of this equation.

$$1.6.7. \text{ Solve } 2^{2x+1} = 3^{x-1}.$$

Taking the natural logarithm on both sides of the equation, we have $(2x + 1) \ln 2 = (x - 1) \ln 3$. Then $(2 \ln 2)x + \ln 2 = (\ln 3)x - \ln 3, (2 \ln 2 - \ln 3)x = -\ln 2 - \ln 3$.

$$x = -\frac{\ln 2 + \ln 3}{2 \ln 2 - \ln 3} \approx 6.23.$$

$$1.6.8. \text{ Solve } e^{2x} - 2e^x - 3 = 0.$$

Let $t = e^x$. Then $t^2 - 2t - 3 = 0, t = -1, 3$. Since $t = e^x > 0, t = 3$, and $x = \ln 3$.

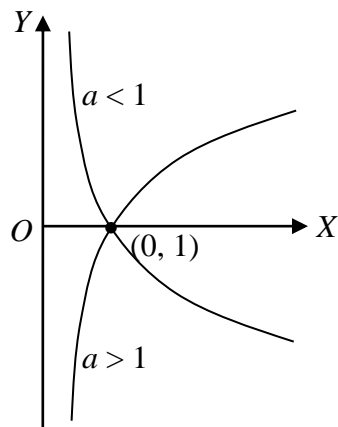
$$1.6.9. \text{ Solve } (\ln x)^2 = \ln(x^2).$$

Since $\ln(x^2) = 2 \ln x$. Let $t = \ln x$. This equation becomes $t^2 = 2t$. The roots of this equation are $t = 0$ and $t = 2$. When $t = \ln x = 0, x = 1$. When $t = \ln x = 2, x = e^2$.

3. The Logarithmic Function

The *logarithmic function* is $y = \log_a x$, where $a > 0$ and $a \neq 1$, called the *base*. Since $x = a^y$, the logarithmic function is the inverse of the exponential function $y = a^x$.

The graph of the logarithmic function is illustrated in the following figure:



4. Properties of the Logarithmic Function

From the graph, we can find a number of important properties of the logarithmic function:

- (a) Domain: $x > 0$.
- (b) Range: $-\infty < y < \infty$.
- (c) Monotonicity: When $a > 1$, the function is increasing; when $a < 1$, this function is decreasing.
- (d) Concavity: When $a < 1$, the graph is concave up; when $a > 1$, the graph is concave down.
- (e) $f(1) = 0$.

Since $a = e^{\ln a}$, $a^x = e^{(\ln a)x}$. We can always express a general exponential function $y = a^x$ as a natural exponential function $y = e^{(\ln a)x}$. By the base-changing formula, we can express a general logarithmic function $y = \log_a x$ as a natural logarithmic function $y = \left(\frac{1}{\ln a}\right) \ln x$ with a constant coefficient.

5. Inverse Trigonometric Functions

Since trigonometric functions are not one-to-one functions, they do not have inverses. However, we can define *restricted* trigonometric functions, i.e., trigonometric function with restricted domain so that these trigonometric functions with specified domain are one-to-one functions, and they have inverses.

Consider the following restricted trigonometric function with specified domain:

$$y = \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

$$y = \cos x, 0 \leq x \leq \pi.$$

$$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

$$y = \sec x, 0 \leq x < \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \leq x < 2\pi.$$

These functions are one-to-one functions. The inverse of these functions are the following:

$$y = \arcsin x = \sin^{-1} x \text{ with domain } [-1, 1] \text{ and range } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

$$y = \arccos x = \cos^{-1} x \text{ with domain } [-1, 1] \text{ and range } 0 \leq y \leq \pi.$$

$$y = \arctan x = \tan^{-1} x \text{ with domain } (-\infty, \infty) \text{ and range } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

$$y = \operatorname{arcsec} x = \sec^{-1} x \text{ with domain } (-\infty, -1), (1, \infty) \text{ and range } 0 \leq x < \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \leq x < 2\pi.$$

Examples

1.6.10. $\sin(\arcsin x) = x$ is true for all x , $-1 \leq x \leq 1$.

$\arcsin(\sin x) = x$ is true only if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Similarly, $\cos(\arccos x) = x$ is true for all x , $-1 \leq x \leq 1$.

$\arccos(\cos x)$ is true only if $0 \leq x \leq \pi$.

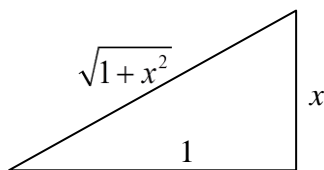
1.6.11. Simplify $\sin(\arctan x)$.

$$\tan u = x, -\frac{\pi}{2} < x < \frac{\pi}{2}. \quad 1 + \tan^2 u = 1 / \cos^2 u = 1 / (1 - \sin^2 u). \quad 1 - \sin^2 u = \frac{1}{1+x^2}.$$

$$\sin^2 u = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2}. \quad \text{Since } \sin u \text{ and } \tan u \text{ have the same sign in this range,}$$

$$\sin u = \frac{x}{\sqrt{1+x^2}}.$$

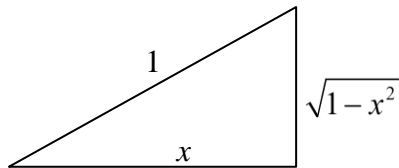
The following diagram illustrates this result:



1.6.12. Simplify $\sin(\operatorname{arcsec} x)$.

$\sec u = x$, $0 \leq x < \frac{\pi}{2}$ or $\pi \leq x < \frac{3\pi}{2}$. $\cos u = 1/x$. $\sin^2 u = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$. Since, in this range of u , $\sin u$ and $\sec u$ have the same sign, $\sin u = \frac{\sqrt{x^2 - 1}}{x}$.

The following diagram illustrates this result:



CHAPTER 2. LIMITS AND DERIVATIVES

§§ 2.2, 2.3, 2.5, 2.6. LIMITS AND CONTINUITY

1. Limits

Definition

Let $y = f(x)$ be a function. If there exists a number L such that $|f(x) - L|$ can be arbitrarily small whenever x is close enough to a value a , then we say that $f(x)$ approaches a *limit* L when x approaches a , denoted by $\lim_{x \rightarrow a} f(x) = L$.

Intuitively, $\lim_{x \rightarrow a} f(x) = L$ if $f(x)$ becomes closer and closer to L when x becomes closer and closer to a .

If $f(x) = c$ is a constant, then $\lim_{x \rightarrow a} f(x) = c$ for any number a . If $f(x) = x$, then $\lim_{x \rightarrow a} f(x) = a$ for any number a .

Limit Laws

Suppose $\lim_{x \rightarrow a} f(x) = L_1$ and $\lim_{x \rightarrow a} g(x) = L_2$. Then

(i) $\lim_{x \rightarrow a} (f(x) \pm g(x)) = L_1 \pm L_2$.

(ii) $\lim_{x \rightarrow a} (f(x)g(x)) = L_1L_2$.

(iii) $\lim_{x \rightarrow a} (f(x)/g(x)) = L_1/L_2$, if $L_2 \neq 0$.

(iv) $\lim_{x \rightarrow a} (f(x))^k = L_1^k$, if the powers are defined.

If the limit laws cannot be used directly, we may have to make some transformations of the expression to find the limit.

Examples

$$2.1.1. \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^3 - 8} = \lim_{x \rightarrow 2} \frac{(x+1)(x-2)}{(x-2)(x^2 + 2x + 4)} = \lim_{x \rightarrow 2} \frac{x+1}{x^2 + 2x + 4} = \frac{3}{12} = \frac{1}{4}.$$

$$2.1.2. \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{2} - \frac{1}{2+h} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2+h-2}{2(2+h)} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{h}{2(2+h)} \right) = \lim_{h \rightarrow 0} \frac{1}{2(2+h)} = \frac{1}{4}.$$

$$2.1.3. \lim_{t \rightarrow 1} \frac{\sqrt{t+3} - 2}{t-1} = \lim_{t \rightarrow 1} \frac{(\sqrt{t+3} - 2)(\sqrt{t+3} + 2)}{(t-1)(\sqrt{t+3} + 2)} = \lim_{t \rightarrow 1} \frac{t-1}{(t-1)(\sqrt{t+3} + 2)} = \lim_{t \rightarrow 1} \frac{1}{\sqrt{t+3} + 2} = \frac{1}{4}.$$

One-Side Limits

If there exists a number L such that $|f(x) - L|$ can be arbitrarily small whenever $x < a$ and it is close enough to a , then we say that $f(x)$ has a *left limit* L when x approaches a^- , denoted by

$$\lim_{x \rightarrow a^-} f(x) = L.$$

If there exist a number L such that $|f(x) - L|$ can be arbitrarily small whenever $x > a$ and it is close enough to a , then we say that $f(x)$ has a *right limit* L when x approaches a^+ , denoted by

$$\lim_{x \rightarrow a^+} f(x) = L.$$

Intuitively, $\lim_{x \rightarrow a^-} f(x) = L$ if $f(x)$ becomes closer and closer to L whenever x becomes closer and closer to a from the left-hand side; $\lim_{x \rightarrow a^+} f(x) = L$ if $f(x)$ becomes closer and closer to L when x becomes closer and closer to a from the right-hand side.

Limit $\lim_{x \rightarrow a} f(x)$ exists if and only if both limits $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ exist and they are equal.

Examples

$$2.1.4. \text{ Let } f(x) = \begin{cases} 3 - x, & x > 0, \\ x - 2, & x < 0. \end{cases}$$

$$\text{Then } \lim_{x \rightarrow 0^+} f(x) = 3, \lim_{x \rightarrow 0^-} f(x) = -2.$$

$$2.1.5. \text{ Let } f(x) = \frac{x^2 - 3x + 2}{|x - 1|}.$$

$$\text{Then } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{(x-1)(x-2)}{x-1} = \lim_{x \rightarrow 1^+} (x-2) = -1, \text{ and}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{(x-1)(x-2)}{1-x} = -\lim_{x \rightarrow 1^+} (x-2) = 1.$$

Limit at Infinity and Infinity as the Limit

Let $y = f(x)$ be a function. If there exists a number L such that $|f(x) - L|$ can be arbitrarily small whenever x is big enough, then we say that $f(x)$ approaches a *limit* L when x approaches infinity, denoted by $\lim_{x \rightarrow \infty} f(x) = L$. Similarly, if there exists a number L such that $|f(x) - L|$ can be arbitrarily small whenever x is small enough, then we say that $f(x)$ approaches a *limit* L when x approaches negative infinity, denoted by $\lim_{x \rightarrow -\infty} f(x) = L$.

Intuitively, $\lim_{x \rightarrow \infty} f(x) = L$ if $f(x)$ becomes closer and closer to L when x becomes bigger and bigger without an upper bound; $\lim_{x \rightarrow -\infty} f(x) = L$ if $f(x)$ becomes closer and closer to L when x becomes smaller and smaller without a lower bound.

If $\lim_{x \rightarrow \infty} f(x) = L$, the graph of $f(x)$ becomes closer and closer to the line $y = L$ when x becomes bigger and bigger; if $\lim_{x \rightarrow -\infty} f(x) = L$, the graph of $f(x)$ becomes closer and closer to the line $y = L$ when x becomes smaller and smaller. In either case, the line $y = L$ is said to be a *horizontal asymptote* of (the graph of) $f(x)$.

If function $y = f(x)$ can be greater than any number whenever x is close enough to a value a , then $f(x)$ approaches infinity when x approaches a , denoted by $\lim_{x \rightarrow a} f(x) = \infty$. If function $y = f(x)$ can be smaller than any number whenever x is close enough to a value a , then $f(x)$ approaches negative infinity when x approaches a , denoted by $\lim_{x \rightarrow a} f(x) = -\infty$.

Intuitively, $\lim_{x \rightarrow a} f(x) = \infty$ if $f(x)$ becomes bigger and bigger without an upper bound when x becomes closer and closer to a ; $\lim_{x \rightarrow a} f(x) = -\infty$ if $f(x)$ becomes smaller and smaller without a lower bound when x becomes closer and closer to a .

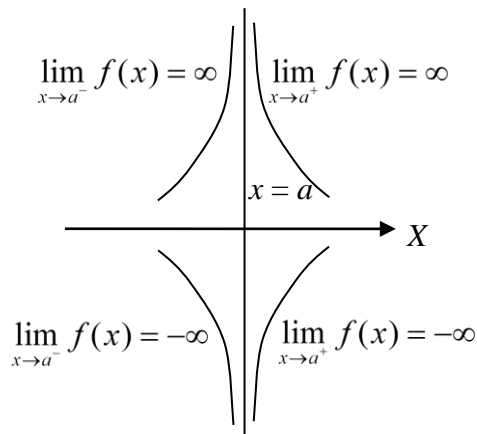
Note that, infinity and negative infinity are NOT numbers. Therefore, if the limit of $f(x)$ when x approaches a is infinity or negative infinity, the limit DOES NOT exist.

In a similar way, we can define infinity and negative infinity as one-side limit:

$$\lim_{x \rightarrow a^-} f(x) = \infty, \lim_{x \rightarrow a^+} f(x) = \infty, \lim_{x \rightarrow a^-} f(x) = -\infty, \text{ or } \lim_{x \rightarrow a^+} f(x) = -\infty.$$

Then $\lim_{x \rightarrow a} f(x) = \infty$ if and only if $\lim_{x \rightarrow a^-} f(x) = \infty$ and $\lim_{x \rightarrow a^+} f(x) = \infty$; $\lim_{x \rightarrow a} f(x) = -\infty$ if and only if $\lim_{x \rightarrow a^-} f(x) = -\infty$ and $\lim_{x \rightarrow a^+} f(x) = -\infty$.

The following figure illustrates these four cases:



In any of these cases, $x = a$ is said to be a *vertical asymptote* of (the graph of) $f(x)$.

Remarks

(a) A function may have at most two horizontal asymptotes, but it may have any number, even infinitely many, of vertical asymptotes.

(b) A horizontal asymptote may have any number, even infinitely many, of intersections with the graph of the function, but a vertical asymptote will never intersect with the graph of the function.

(c) Asymptotes are lines. We may say $x = 2$ is a vertical asymptote, or $y = -1$ is a horizontal asymptote, but 2 and -1 are numbers, NOT asymptotes.

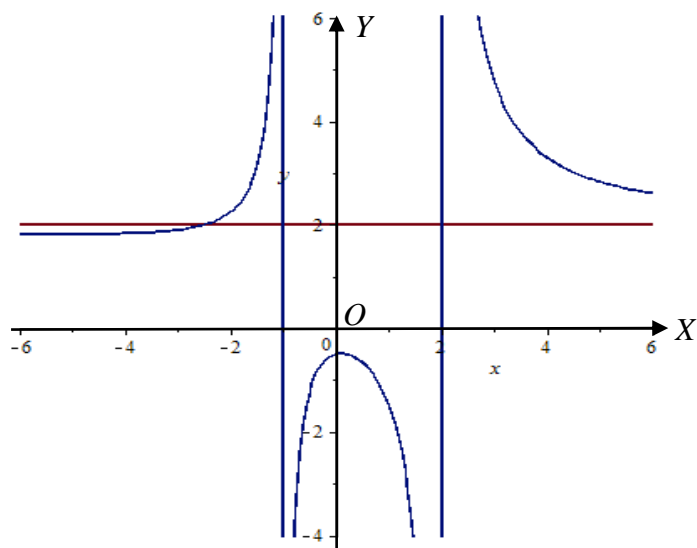
Examples

2.1.6. Consider function $y = \frac{2x^2 + 1}{x^2 - x - 2}$.

Since $\lim_{x \rightarrow \infty} \frac{2x^2 + 1}{x^2 - x - 2} = \lim_{x \rightarrow -\infty} \frac{2x^2 + 1}{x^2 - x - 2} = 2$, $y = 2$ is a horizontal asymptote.

Since $\lim_{x \rightarrow -1^+} \frac{2x^2 + 1}{x^2 - x - 2} = -\infty$, $\lim_{x \rightarrow 2^-} \frac{2x^2 + 1}{x^2 - x - 2} = -\infty$, $\lim_{x \rightarrow -1^-} \frac{2x^2 + 1}{x^2 - x - 2} = \infty$ and $\lim_{x \rightarrow 2^+} \frac{2x^2 + 1}{x^2 - x - 2} = \infty$, $x = -1$ and $x = 2$ are vertical asymptotes.

The graph of this function looks like the following:



2.1.7. Consider function $y = \frac{x-2}{x^2-x-2}$.

Since $\frac{x-2}{x^2-x-2} = \frac{x-2}{(x-2)(x+1)} = \frac{1}{x+1}$, this function is $y = \frac{1}{x+1}$, defined for $x \neq -1$, and $x \neq 2$.

Since $\lim_{x \rightarrow -1^+} \frac{1}{x+1} = \infty$, $\lim_{x \rightarrow -1^-} \frac{1}{x+1} = -\infty$, and $\lim_{x \rightarrow \infty} \frac{1}{x+1} = 0$, $\lim_{x \rightarrow -\infty} \frac{1}{x+1} = 0$, this function has a vertical asymptote $x = -1$, and a horizontal asymptote $y = 0$. Note that $x = 2$ is not a vertical asymptote.

2.1.8. Consider function $f(x) = \frac{\sqrt{x^2+1}}{2x}$.

Since $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}}{2x} = \frac{1}{2}$, and $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+1}}{2x} = -\frac{1}{2}$ (note that when $x < 0$, $x = -\sqrt{x^2}$), the graph of this function has two horizontal asymptotes $y = \frac{1}{2}$, and $y = -\frac{1}{2}$.

Since $\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2+1}}{2x} = \infty$ and $\lim_{x \rightarrow 0^-} \frac{\sqrt{x^2+1}}{2x} = -\infty$, the graph of this function has a vertical asymptote $x = 0$.

2. Continuity

Definition

A function $f(x)$ is said to be *continuous* at $x = a$ if $\lim_{x \rightarrow a} f(x) = f(a)$. If $f(x)$ is continuous at every point in an interval I , then it is *continuous in I*.

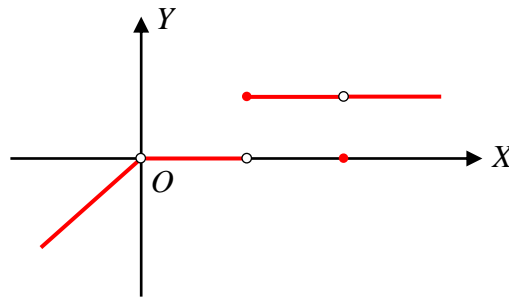
By definition, $f(x)$ is not continuous at $x = a$ if (i) $\lim_{x \rightarrow a} f(x)$ does not exist, or (ii) a is not in the domain of $f(x)$, or (iii) $\lim_{x \rightarrow a} f(x) \neq f(a)$.

Intuitively, $f(x)$ is continuous at $x = a$ if the graph of $f(x)$ does not "break" at this point.

Examples

2.1.9. Consider function $f(x) = \begin{cases} x & x < 0 \\ 0 & 0 < x < 1 \\ 1 & 1 \leq x < 2 \\ 0 & x = 2 \\ 1 & x > 2 \end{cases}$.

This function is not continuous at $x = 0$ because $x = 0$ is not in the domain of this function although $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 0$. It is not continuous at $x = 1$ because $\lim_{x \rightarrow 1^-} f(x) = 0 \neq \lim_{x \rightarrow 1^+} f(x) = 1$. The limit $\lim_{x \rightarrow 1} f(x)$ does not exist. It is not continuous at $x = 2$ because $\lim_{x \rightarrow 2} f(x) = 1$ but $f(2) = 0$. The graph of this function looks like the following:



2.1.10. If $f(x) = \begin{cases} x^2 - ax & x > 2 \\ x + a & x \leq 2 \end{cases}$ is continuous at $x = 2$, find a .

$\lim_{x \rightarrow 2^+} (x^2 - ax) = 4 - 2a$, $\lim_{x \rightarrow 2^-} (x + a) = 2 + a$. Let $4 - 2a = 2 + a$. $3a = 2$, $a = \frac{2}{3}$.

2.1.11. Let

$$f(x) = \begin{cases} a - x, & x < -2 \\ 2 + bx, & -2 \leq x \leq 2 \\ ax^2 + 8b, & x > 2 \end{cases}.$$

For what values of the constants a and b is function $f(x)$ continuous for all real numbers?

This function is continuous for all values $x \neq -2$ and $x \neq 2$. If this function is continuous at $x = -2$, we must have $\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} (a - x) = a + 2 = \lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} (2 + bx) = 2 - 2b$, and

$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2 + bx) = 2 + 2b = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (ax^2 + 8b) = 4a + 8b$. Then we have a system of two linear equations: $a + 2 = 2 - 2b$ and $2 + 2b = 4a + 8b$. Solving this system of equations, we have $a = 2$, $b = -1$.

The Extreme Value Theorem. If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum and an absolute minimum in this interval.

3. Limits and Continuity of Polynomials, Rational Functions, Trigonometric Functions, Exponential Functions, and Logarithmic functions

Polynomial Functions

Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$ be a polynomial, $c_n > 0$. Then $\lim_{x \rightarrow \infty} P(x) = \infty$,

$\lim_{x \rightarrow -\infty} P(x) = \infty$ if n is even, and $\lim_{x \rightarrow -\infty} P(x) = -\infty$ if n is odd. The graphs of polynomial functions do not have horizontal asymptotes.

Polynomial functions are continuous on the entire real axis. The graphs of polynomial functions do not have vertical asymptotes.

Rational Functions

Let $f(x) = \frac{M(x)}{N(x)}$, where $M(x)$ and $N(x)$ are polynomials, be a rational function.

A rational function $f(x) = \frac{M(x)}{N(x)}$ is continuous at every real number except the zeros of $N(x)$.

If $x = a$ is a zero of $N(x)$ but not a zero of $M(x)$, then $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ are ∞ or $-\infty$, and $x = a$ is a vertical asymptote of this function.

Assume the leading term of $M(x)$ is $a_m x^m$ and the leading term of $N(x)$ is $b_n x^n$.

If $m < n$, $\lim_{x \rightarrow \infty} \frac{M(x)}{N(x)} = \lim_{x \rightarrow -\infty} \frac{M(x)}{N(x)} = 0$. In this case, this function has a horizontal asymptote $y = 0$, i.e., the x -axis.

If $m = n$, $\lim_{x \rightarrow \infty} \frac{M(x)}{N(x)} = \lim_{x \rightarrow -\infty} \frac{M(x)}{N(x)} = \frac{a_m}{b_n}$. In this case, this function has a horizontal asymptote $y = \frac{a_m}{b_n}$.

If $m > n$, $\lim_{x \rightarrow \pm\infty} \frac{M(x)}{N(x)} = \lim_{x \rightarrow \pm\infty} \frac{a_m x^{m-n}}{b_n}$, which can be infinity or negative infinity depending on the signs of a_m and b_n , and x approaching infinity or negative infinity. In this case, this function does not have a horizontal asymptote.

Trigonometric Functions

Functions $y = \sin x$ and $y = \cos x$ are defined and continuous on the entire real axis. The graphs of $\sin x$ and $\cos x$ do not have vertical asymptotes or horizontal asymptotes.

An important limit about the sine function, to be proved by the squeeze theorem, is

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

The following properties of functions $\tan x$, $\cot x$, $\sec x$, and $\csc x$, can easily be seen from the graphs of these functions:

Function $y = \tan x$ is continuous at all real numbers except $x = n\pi + \frac{\pi}{2}$, where n is any integer.

Lines $x = n\pi + \frac{\pi}{2}$, where n is an integer, are vertical asymptotes of the graph of function $\tan x$.

$\lim_{x \rightarrow (n\pi + \pi/2)^-} \tan x = \infty$, $\lim_{x \rightarrow (n\pi + \pi/2)^+} \tan x = -\infty$. The graph of function $\tan x$ does not have horizontal asymptotes.

Function $y = \cot x$ is continuous at all real numbers except $x = n\pi$, where n is any integer.

Lines $x = n\pi$, where n is an integer, are vertical asymptotes of the graph of function $\cot x$.

$\lim_{x \rightarrow n\pi^+} \cot x = \infty$, $\lim_{x \rightarrow n\pi^-} \cot x = -\infty$. The graph of function $\cot x$ does not have horizontal asymptotes.

Function $y = \sec x$ is continuous at all real numbers except $x = n\pi + \frac{\pi}{2}$, where n is any integer.

Lines $x = n\pi + \frac{\pi}{2}$, where n is an integer, are vertical asymptotes of the graph of function $\sec x$.

$\lim_{x \rightarrow (2n\pi + \pi/2)^-} \sec x = \lim_{x \rightarrow (2n\pi - \pi/2)^+} \sec x = \infty$, $\lim_{x \rightarrow ((2n+1)\pi + \pi/2)^-} \sec x = \lim_{x \rightarrow ((2n+1)\pi - \pi/2)^+} \sec x = -\infty$. The graph of function $\sec x$ does not have horizontal asymptotes.

Function $y = \csc x$ is continuous at all real numbers except $x = n\pi$, where n is any integer.

Lines $x = n\pi$, where n is an integer, are vertical asymptotes of the graph of function $\csc x$.

$\lim_{x \rightarrow 2n\pi^+} \csc x = \lim_{x \rightarrow (2n+1)\pi^-} \csc x = \infty$, $\lim_{x \rightarrow 2n\pi^-} \csc x = \lim_{x \rightarrow (2n+1)\pi^+} \csc x = -\infty$. The graph of function $\csc x$ does not have horizontal asymptotes.

Exponential Functions

Let $y = a^x$, $a > 0$, $a \neq 1$, be an exponential function.

Exponential functions are continuous on the entire real axis. The graphs of exponential functions do not have vertical asymptotes.

When $a > 1$, $\lim_{x \rightarrow \infty} a^x = \infty$, $\lim_{x \rightarrow -\infty} a^x = 0$; when $a < 1$, $\lim_{x \rightarrow \infty} a^x = 0$, $\lim_{x \rightarrow -\infty} a^x = \infty$. In either case, $y = 0$ is a horizontal asymptote.

When $a > 1$, exponential function increases faster than any power function, namely, when $a > 1$,

$$\lim_{x \rightarrow \infty} \frac{x^k}{a^x} = 0 \text{ for any } k.$$

The number e used as the base of natural exponential function is defined to be

$$e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = 2.71828\dots$$

An important limit about the natural exponential function is

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1.$$

Logarithmic Functions

The logarithmic function is continuous everywhere in its domain, i.e., $x > 0$.

When $a > 1$, $\lim_{x \rightarrow \infty} \log_a x = \infty$, $\lim_{x \rightarrow 0^+} \log_a x = -\infty$; when $a < 1$, $\lim_{x \rightarrow \infty} \log_a x = 0$, $\lim_{x \rightarrow 0^+} \log_a x = \infty$. In either case, $x = 0$ is a vertical asymptote. The graphs of logarithmic functions do not have horizontal asymptotes.

Although, when $a > 1$, logarithmic function approaches infinity when x approaches infinity, it increases slower than any power function x^k , $k > 0$. Namely, when $a > 1$ and $k > 0$,

$$\lim_{x \rightarrow \infty} \frac{\log_a x}{x^k} = 0. \text{ In particular, } \lim_{x \rightarrow \infty} \frac{\ln x}{x^k} = 0 \text{ for any } k > 0.$$

An important limit about the natural logarithmic function is

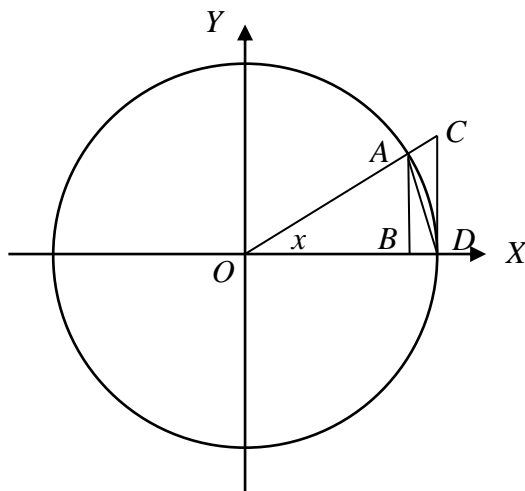
$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1.$$

4. The Squeeze Theorem

Suppose functions $f(x)$, $g(x)$, and $h(x)$ are defined and continuous in an interval (b, c) , and $f(x) \geq h(x) \geq g(x)$ for every x in (b, c) . If $b < a < c$, and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = L$, then $\lim_{x \rightarrow a} h(x) = L$.

Example 2.1.12. Prove $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ by squeeze theorem.

In the following figure, circle O is a unit circle. Then $AB = \sin x$, $CD = \tan x$ and the length of the arc AD is x . The area of $\triangle OCD$ is $A_1 = \frac{1}{2} CD$, the area of the sector OAD is $A_2 = \frac{1}{2} x$, and the area $\triangle ABD$ is $A_3 = \frac{1}{2} AB$. Since $A_3 < A_2 < A_1$, we see that $\sin x < x < \tan x$ when $x < \frac{\pi}{2}$.



Hence, $1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$. Taking the reciprocal, $1 > \frac{\sin x}{x} > \cos x$. Since $\lim_{x \rightarrow 0} \cos x = 1$, we have $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ by the squeeze theorem. Note that the angle x is measured in radians, but not degrees.

5. Intermediate-Value Theorem

When a function $f(x)$ is continuous in $[a, b]$, $f(a) \neq f(b)$, then for every value y_0 between $f(a)$ and $f(b)$, there is c between a and b such that $f(c) = y_0$.

In particular, if $f(a)$ and $f(b)$ have different signs, then there must be a value c , $a < c < b$, such that $f(c) = 0$.

Using this result, we have the following procedure to solve inequalities of the form $f(x) > 0$ or $f(x) < 0$: Find the roots of the equation $f(x) = 0$. These roots subdivide the domain of $f(x)$ into a number of intervals, such that, in each of these intervals, $f(x)$ is continuous. Since $f(x)$ has the same sign for all x in each of these intervals, we can find the intervals where $f(x) > 0$ or $f(x) < 0$.

Examples

2.1.13. Consider polynomial $P(x) = x^3 - 2x^2$.

Let $P(x) = x^3 - 2x^2 = x^2(x - 2) = 0$. The zeros of $P(x)$ are $x = 0$ and $x = 2$, which subdivide the domain of $P(x)$, i.e., the x -axis, into intervals $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$. Function $P(x)$ does not change sign in each of these intervals. Since $P(-1) < 0$, $P(1) < 0$, and $P(3) > 0$, we see that, in interval $(-\infty, 0)$ and $(0, 2)$, $P(x) < 0$, and, in interval $(2, \infty)$, $P(x) > 0$.

2.1.14. Consider rational function $f(x) = \frac{(x+2)(x-5)}{x-1}$.

The domain of this function consists of intervals $(-\infty, 1)$ and $(1, \infty)$. The zeros of polynomial $(x+2)(x-5)$ are $x = -2$ and $x = 5$, which subdivide the domain of this function into four subintervals $(-\infty, -2)$, $(-2, 1)$, $(1, 5)$ and $(1, \infty)$. In each of these intervals, function $f(x)$ does not change sign.

Since $f(-3) = -2 < 0$, $f(0) = 10 > 0$, $f(2) = -12 < 0$, and $f(6) = \frac{8}{5} > 0$, we have

$f(x) < 0$ when $x < -3$, $1 < x < 5$, and $f(x) > 0$ when $-3 < x < 1$ or $x > 5$.

2.1.15. Solve inequality $\frac{1}{x+1} < \frac{1}{5}$.

This inequality is equivalent to $\frac{1}{x+1} - \frac{1}{5} = \frac{5-x-1}{5(x+1)} = \frac{4-x}{x+1} < 0$.

The left-hand side is not defined at $x = -1$. It is zero when $x = 4$.

The function $\frac{4-x}{x+1}$ does not change sign in each of the intervals $(-\infty, -1)$, $(-1, 4)$ and $(4, \infty)$.

Since $f(-2) = -6 < 0$, $f(0) = 4 > 0$, and $f(5) = -\frac{1}{6} < 0$. The solution of this inequality is $x < -1$ or $x > 5$.

§§ 2.1, 2.7, 2.8. RATE OF CHANGE AND DERIVATIVES

1. Average Rate of Change

Let $y = f(x)$ be a function. When x changes from $x = a$ to $x = b$, $b - a$ is the *change* (also called the *increment*) of the variable, $f(b) - f(a)$ is the *change* (also called the *increment*) of the function, and $r(f; a, b) = \frac{f(b) - f(a)}{b - a}$ is the *average rate of change* of this function from $x = a$ to $x = b$.

If $r(f; a, b) > 0$, function $f(x)$ *increases* from $x = a$ to $x = b$, and, if $r(f; a, b) < 0$, function $f(x)$ *decreases* from $x = a$ to $x = b$. If $|r(f; a, b)|$ is big, this function changes quickly from $x = a$ to $x = b$, and, if $|r(f; a, b)|$ is small, this function changes slowly from $x = a$ to $x = b$.

Geometric interpretation: The average rate of change of function $y = f(x)$ from $x = a$ to $x = b$ is the slope of the secant line joining two points $(a, f(a))$ and $(b, f(b))$ on the graph of this function.

Physics interpretation: If $x = s(t)$ is the position function of the x coordinate of a particle moving along the x -axis, then the average rate of change of this function from $t = a$ to $t = b$ is the *average velocity* of the particle.

Example

2.2.1. Consider function $f(x) = 2x - 0.1x^2$.

$$f(2) = 3.6, f(5) = 7.5.$$

The average rate of change of this function from $x = 2$ to $x = 5$ is

$$r(f; a, b) = \frac{7.5 - 3.6}{5 - 2} = 1.3.$$

Geometric interpretation: The slope of the secant line joining two points $(2, 3.6)$ and $(5, 7.5)$ on the graph of $y = f(x)$ is 1.3. Hence, the equation of the secant line is $y = 1.3(x - 2) + 3.6$, or $y = 1.3x + 1$.

Physics interpretation: Write this function as $x = 2t - 0.1t^2$, where x is in meters, and t is in seconds, interpreted as the position function of a particle moving along the x -axis. Then the position of this particle at time $t = 2$ is $x(2) = 3.6$ meters right to the origin, the position of this particle at time $t = 5$ is $x(5) = 7.5$ meters right to the origin, and the average velocity of the particle between $t = 2$ and $t = 5$ is 1.3 m/sec.

2. Instantaneous Rate of Change – the Derivative

Definition

When b approaches a , the limit of the average rate of change of function $y = f(x)$ from $x = a$ to $x = b$, if it exists, is **defined to be** the *instantaneous rate of change* of this function at $x = a$. This is also called the *derivative* of this function at $x = a$, denoted by $f'(a)$:

$$f'(a) = \lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a}.$$

If write $b = a + h$, then $b - a = h$, and

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

Let $y = f(x)$ be a function defined in an interval around a point $x = a$. If $f'(a) > 0$, this function is increasing in an interval around $x = a$. In this case, we say that $f(x)$ is *increasing at $x = a$* . If $f'(a) < 0$, this function is decreasing in an interval around $x = a$. In this case, we say that $f(x)$ is *decreasing at $x = a$* . If $|f'(a)|$ is small, this function changes (increases or decreases) slowly in an interval around the value a , and, if $|f'(a)|$ is big, this function changes (increases or decreases) quickly in an interval around the value $x = a$.

Since $f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$, when h is small, $f(a + h) - f(a) \approx f'(a)h$. In other words, when h is small, we can use $f'(a)h$ to be an approximation of the increment of the function.

Geometric interpretation: The derivative $f'(a)$ is the slope of the tangent line at the point $(a, f(a))$ on the graph of function $y = f(x)$. In particular, since the slope of a horizontal line is 0, the graph of function $f(x)$ has a horizontal tangent line at $x = a$ if and only if $f'(a) = 0$.

Physics interpretation: If $x = s(t)$ is the position function of the x coordinate of a particle moving along the x -axis, then the derivative of this function at $t = a$ is the *instantaneous velocity* of the particle at $t = a$.

The Derivative Function

When a takes different values in the domain of a function $y = f(x)$, the derivative $f'(a)$ also changes. Then we may regard the values of the derivatives with different values of x as a new function "derived" from the original function. This is the *derivative function*, or simply the *derivative* of the function, denoted by y' (to specify the variable, we may also write y'_x), or $f'(x)$, or $\frac{dy}{dx}$, or $\frac{d}{dx} f(x)$.

The *derivative of a function at a point* is a number, while the *derivative of a function* is a function.

If $f'(x) > 0$ for all values of x , $a < x < b$, then $f(x)$ is *increasing in interval* (a, b) ; if $f'(x) < 0$ for all values of x , $a < x < b$, then $f(x)$ is *decreasing in interval* (a, b) .

Approximation of the Derivative by Average Rate of Change

Since the derivative is the limit of the average rate of change, we can use an average rate of change of a function $y = f(x)$ in a small interval around a point $x = a$ to estimate the derivative $f'(a)$.

Let h be a small positive value. Then we have

$$r(f; a, a + h) = \frac{f(a + h) - f(a)}{h} \approx f'(a). \quad (\text{The forward formula})$$

$$r(f; a - h, a) = \frac{f(a) - f(a - h)}{h} \approx f'(a). \quad (\text{The backward formula})$$

$$r(f; a - h, a + h) = \frac{f(a + h) - f(a - h)}{h} \approx f'(a). \quad (\text{The three-point formula})$$

Find the derivative of a function by Definition

We can find the derivative of some simple functions directly by the definition.

Examples

2.2.2. Find the derivative of function $f(x) = 2x - 0.1x^2$ at $x = 4$ by definition.

$$f(4) = 8 - 1.6 = 6.4.$$

$$y' = \lim_{h \rightarrow 0} \frac{(2(4 + h) - 0.1(4 + h)^2) - 6.4}{h} = \lim_{h \rightarrow 0} \frac{(8 + 2h) - 0.1(16 + 8h + h^2) - 6.4}{h} = \lim_{h \rightarrow 0} (1.2 - 0.1h)$$

= 1.2.

The slope of the tangent line of the graph of $f(x)$ at $(4, 6.4)$ is $m = 1.2$. The equation of the tangent line is $y = 1.2(x - 4) + 6.4$, or $y = 1.2x + 1.6$.

If this function is the position function $x = 2t - 0.1t^2$, then the instantaneous velocity of this particle at $t = 4$ is 1.2.

2.2.3. Find the derivative of the function $y = \frac{1}{2x-1}$ at $x = 2$ by definition.

$$\begin{aligned} y'(2) &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{2(2+h)-1} - \frac{1}{2 \times 2 - 1} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{3 - (3+2h)}{3(3+2h)} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-2h}{3(3+2h)} \right) \\ &= -2 \lim_{h \rightarrow 0} \frac{1}{3(3+2h)} = -\frac{2}{9}. \end{aligned}$$

2.2.4. Find the derivative of function $y = \frac{1}{\sqrt{2x+1}}$ by definition.

$$\begin{aligned} y' &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{\sqrt{2(x+h)+1}} - \frac{1}{\sqrt{2x+1}} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \frac{\sqrt{2x+1} - \sqrt{2(x+h)+1}}{\sqrt{2(x+h)+1}\sqrt{2x+1}} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{(\sqrt{2x+1} - \sqrt{2(x+h)+1})(\sqrt{2x+1} + \sqrt{2(x+h)+1})}{\sqrt{(2(x+h)+1)(2x+1)}(\sqrt{2x+1} + \sqrt{2(x+h)+1})} \\ &= \lim_{x \rightarrow 0} \frac{1}{h} \frac{(2x+1) - (2x+2h+1)}{\sqrt{(2(x+h)+1)(2x+1)}(\sqrt{2x+1} + \sqrt{2(x+h)+1})} \\ &= -2 \lim_{x \rightarrow 0} \frac{1}{\sqrt{(2(x+h)+1)(2x+1)}(\sqrt{2x+1} + \sqrt{2(x+h)+1})} \\ &= -2 \lim_{x \rightarrow 0} \frac{1}{\sqrt{(2x+1)(2x+1)}(\sqrt{2x+1} + \sqrt{2x+1})} = -\frac{2}{2(2x+1)\sqrt{2x+1}} = -(2x+1)^{-3/2}. \end{aligned}$$

Cases when the Derivative Does Not Exist

(a) If a is not in the domain of $f(x)$, $f(a)$ does not exist, the derivative cannot be defined.

(b) If a is in the domain of $f(x)$, but $\lim_{h \rightarrow 0} f(a+h) \neq f(a)$, then the limit $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ does not exist. Therefore, if the function is not continuous at $x = a$, $f'(a)$ does not exist.

In particular, if $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ is infinity or negative infinity. Then the derivative does not exist, and the graph of the function has a vertical tangent line at $x = a$.

(c) If $f(x)$ is continuous at $x = a$ but $\lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h} \neq \lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h}$, then $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ does not exist. In this case, the graph of the function has a "sharp corner" at $x = a$.







3. Higher Derivatives

The second derivative of a function $y = f(x)$ is the derivative of the derivative of $f(x)$, denoted by y'' or $f''(x)$ or $\frac{d^2 y}{dx^2}$ or $\frac{d^2}{dx^2} f(x)$.

The second derivative of a displacement function is the acceleration function.

Geometric Meaning: If $f''(x) > 0$, $f'(x)$ is increasing, and the graph of the function is concave up; if $f''(x) < 0$, then $f'(x)$ is decreasing, and the graph of the function is concave down.

The following table is an illustration of the relation among the properties of a function and its derivatives:

$f(x)$	$f'(x)$	$f''(x)$
	+	
	-	
		+
		-

In a similar way, we can find the third, the fourth, ..., the n -th derivatives of a function. The n -th derivative is denoted by any of the following notations:

$$y^{(n)}(x) = f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n} f(x).$$

CHAPTER 3. DIFFERENTIATION RULES

§ 3.1, 2.3, 3.3. BASIC FORMULAS

1. Basic Formulas

$(cf(x))' = cf'(x)$ if $f'(x)$ exists, (the constant multiplication rule)

$(f(x) \pm g(x))' = f'(x) \pm g'(x)$ if $f'(x)$ and $g'(x)$ exist, (the sum / difference rule)

$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$ (the product rule)

$(f(x)/g(x))' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ (the quotient rule)

2. Formulas of Basic Functions

$(c)' = 0,$

$(x^n)' = nx^{n-1},$ (the power rule)

$(\sin x)' = \cos x,$

$(\cos x)' = -\sin x,$

$(\tan x)' = \sec^2 x,$

$(\cot x)' = -\csc^2 x,$

$(e^x)' = e^x,$

$(\ln x)' = \frac{1}{x}.$

3. Examples

$$3.1.1. y = \log_a x.$$

$$\text{Since } \log_a x = \frac{1}{\ln a} \ln x, (\log_a x)' = \frac{1}{\ln a} (\ln x)' = \frac{1}{(\ln a)x}.$$

$$3.1.2. y = 3x^3 + 2\sqrt{x} - \log_2 x.$$

$$y' = 9x^2 + \frac{1}{\sqrt{x}} + \frac{1}{(\ln 2)x}.$$

$$3.1.3. y = \frac{2t^2 - 1}{\sqrt{t}}.$$

$$y' = \frac{d}{dt} \left(\frac{2t^2 - 1}{\sqrt{t}} \right) = \frac{d}{dt} (2t^{3/2} - t^{-1/2}) = 2 \left(\frac{3}{2} t^{1/2} \right) + \frac{1}{2} t^{-3/2} = 3\sqrt{t} + \frac{1}{2t\sqrt{t}}.$$

$$3.1.4. y = x^3 \ln x.$$

$$y' = 3x^2 \ln x + x^2.$$

3.1.5. Find the value(s) of x where the graph of function $y = \frac{e^x}{x^2}$ has a horizontal tangent line.

$$y' = \frac{e^x x^2 - 2xe^x}{x^4} = \frac{e^x(x-2)}{x^3}.$$

Let $y' = 0$. $x = 2$. The graph of this function has a horizontal tangent line at the point where $x = 2$.

3.1.6. Find the equation of the tangent line of the graph of function $y = \frac{x^2}{\ln x}$ at the point (e, e^2) .

$$y' = \frac{2x \ln x - x^2(1/x)}{(\ln x)^2} = \frac{x(2 \ln x - 1)}{(\ln x)^2}. \quad y'(e) = e.$$

The equation of the tangent line is $y = e(x - e) + e^2$, or $y = ex$.

$$3.1.7. y = \frac{xe^x}{x^2 + 1}.$$

$$y' = \frac{(e^x + xe^x)(x^2 + 1) - xe^x(2x)}{(x^2 + 1)^2} = \frac{x^2e^x + e^x + x^3e^x + xe^x - 2x^2e^x}{(x^2 + 1)^2} = \frac{e^x(x^3 - x^2 + x + 1)}{(x^2 + 1)^2}.$$

§ 3.4. THE CHAIN RULE

The chain rule is used to find the derivative of a composite function. Let $y = f(u)$ and $u = g(x)$.

Then y is a composite function of x : $y = f(g(x)) = (f \circ g)(x)$. The derivative of this composite

function is calculated by $\frac{d}{dx}(f \circ g)(x) = \frac{d}{du}f(u) \frac{d}{dx}g(x)$ or, simply, $y'_x = y'_u u'_x$. In the

simplified version of the formula, remember that y'_x on the left-hand side is the derivative of y as the composite function of x , and y'_u on the right-hand side is the derivative of y as a function of u .

Recall that, when we want to evaluate a function $y = f(x)$ with a given value of x , we plug this value into this function, and, after a sequence of operations, a value of y is reached. Now look at the operation used at the last step. If the last operation is an addition, subtraction, multiplication or division, use the sum rule, difference rule, product rule or the quotient rule, respectively. Otherwise, decompose this function into a composition of two functions, and use the chain rule.

Examples

3.2.1. Some values of functions $f(x)$ and $g(x)$, and their derivatives $f'(x)$ and $g'(x)$ are given in the following table:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	6	-2	5	3
3	1	2	2	7
5	-3	4	3	4

If $y = h(x) = (f \circ g)(x)$, find $h(3)$ and $h'(3)$?

Let $u = g(x)$. When $x = 3$, $u = 2$. Then $h(x) = f(g(x)) = f(u)$. $h(3) = f(g(3)) = f(2) = 6$.

Since $y = f(g(x)) = f(u)$, $y'_u = f'(2) = -2$, and $u'_x = g'(3) = 7$. By the chain rule, $y'_x = y'_u u'_x$, $h'(3) = y'_x = f'(2)g'(3) = -14$.

3.2.2. $y = e^{kx}$.

Let $u = kx$. Then $y = e^u$. Since $u'_x = k$, and $y'_u = e^u$. we have $y'_x = ke^u = ke^{kx}$.

Since $a = e^{\ln a}$, the general exponential function $y = a^x$ can be written as $y = e^{(\ln a)x}$. Let $u = (\ln a)x$. Then $y = e^u$. By the chain rule, we have

$$(a^x)' = (\ln a)e^{(\ln a)x} = (\ln a)a^x.$$

$$3.2.3. \quad y = \ln(-x), \quad x < 0.$$

Let $u = -x$. Then $y = \ln u$. Since $u'_x = -1$, and $y'_u = \frac{1}{u}$, we have $y' = (-1)\left(\frac{1}{u}\right) = -\left(-\frac{1}{x}\right) = \frac{1}{x}$.

Since the derivative of the function $y = \ln x$, $x > 0$, is also $\frac{1}{x}$, the derivative of the function

$$y = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}, \text{ defined for all } x \neq 0, \text{ is } y' = \frac{1}{x}. \text{ Since } |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}, \text{ this function can}$$

be written as $y = \ln|x|$. In other words, the derivative of function $y = \ln|x|$ is $y' = \frac{1}{x}$.

$$3.2.4. \quad y = \sqrt{3x^2 - 2x + 1}.$$

Let $u = 3x^2 - 2x - 1$. Then $y = \sqrt{u}$. Since $u'_x = 6x - 2$, and $y'_u = \frac{1}{2\sqrt{u}}$, we have

$$y'_x = (6x - 2) \left(\frac{1}{2\sqrt{u}} \right) = \frac{3x - 1}{\sqrt{3x^2 - 2x + 1}}.$$

$$3.2.5. \quad y = \ln(e^{\sqrt{x}} + 1).$$

Let $u = e^{\sqrt{x}} + 1$. Then $y = \ln u$. To find u'_x , let $v = \sqrt{x}$. Then $u = e^v = e^{\sqrt{x}}$. We have

$$u'_x = u'_v v'_x = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}. \text{ Since } y'_u = \frac{1}{u} = \frac{1}{e^{\sqrt{x}} + 1}, \quad y' = \frac{1}{e^{\sqrt{x}} + 1} \left(\frac{e^{\sqrt{x}}}{2\sqrt{x}} \right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}(e^{\sqrt{x}} + 1)}.$$

3.2.6. For which value(s) of x , does the graph of the function $y = x^2\sqrt{1-x^2}$ have a horizontal tangent line.

Since the last operation is a product, use the product rule first. $(x^2)' = 2x$.

$$y'_x = 2x\sqrt{1-x^2} + x^2(\sqrt{1-x^2})'.$$

To find the derivative of $z = \sqrt{1-x^2}$, let $u = 1-x^2$. Then $z = \sqrt{u}$. Since $z'_u = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{1-x^2}}$,

$$u'_x = -2x, \quad z'_x = z'_u u'_x = \frac{1}{2\sqrt{1-x^2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

$$\text{Finally, } y'_x = 2x\sqrt{1-x^2} - \frac{x^3}{\sqrt{1-x^2}} = \frac{2x(1-x^2) - x^3}{\sqrt{1-x^2}} = \frac{2x - 3x^3}{\sqrt{1-x^2}}.$$

Let $y' = 0$. Then $2x - 3x^3 = x(2 - 3x^2) = 0$, $x = 0$, $x = \pm\sqrt{\frac{2}{3}}$. The graph of this function has a horizontal tangent line when $x = 0$ or $x = \pm\sqrt{\frac{2}{3}}$.

$$3.2.7. \quad y = e^{\sin x}.$$

Let $u = \sin x$. Then $y = e^u$. The derivative is $y'_x = e^u \cos x = e^{\sin x} \cos x$.

$$3.2.8. \quad y = \sqrt{2\sin x + 4\cos x}. \quad \text{Find } y'(0).$$

Let $u = 2\sin x + 4\cos x$. Then $y = \sqrt{u}$.

$$y'_u = \frac{1}{2\sqrt{u}}. \quad \text{At } x = 0, u = 4. \quad y'_u(4) = \frac{1}{4}.$$

$$u'_x = 2\cos x - 4\sin x. \quad u'_x(0) = 2.$$

$$y'_x(0) = y'_u(4) u'_x(0) = \frac{1}{2}.$$

§ 3.5. IMPLICIT DIFFERENTIATION

1. Implicit Functions Defined by Equations

Let $F(x, y) = G(x, y)$ be a two-variable equation. Let (x_0, y_0) be a solution to this equation, i.e., $F(x_0, y_0) = G(x_0, y_0)$. If, in a small disk around point (x_0, y_0) , for every value x^* , there is a unique value y^* such that $F(x^*, y^*) = G(x^*, y^*)$, i.e., the equation $F(x^*, y) = G(x^*, y)$ has unique root $y = y^*$ in this disk, then let $y^* = f(x^*)$. In this way, we define a function $y = f(x)$ in an interval around the point x_0 . This function is said to be an *implicit function defined by the equation* $F(x, y) = G(x, y)$ *around the point* (x_0, y_0) . We also say that function $f(x, y)$ is *implicitly defined by the equation* $F(x, y) = G(x, y)$ *around the point* (x_0, y_0) .

2. Find the Derivative of Implicit Functions

The derivative of an implicit function $y = f(x)$ defined by an equation $F(x, y) = G(x, y)$ can be found by the chain rule. Since $y = f(x)$, both sides of $F(x, y) = G(x, y)$ are functions of x . Take the derivative on both sides of this equation with respect to x , and use the chain rule regarding y as an intermediate variable. We have an equation with x , y and y' . Solve this equation for y' . We can find the derivative of this implicit function. Note that the derivative of an implicit function obtained by implicit differentiation is usually expressed as a function of both x and y .

Examples

3.3.1. Suppose a function $y = f(x)$ is defined implicitly by the equation $x^3 + y^3 = 6xy$. Find the derivative $f'(x)$.

Take the derivative with respect to x on both sides: $3x^2 + 3y^2y' = 6(y + xy')$.

$$\text{Solve for } y': y' = \frac{2y - x^2}{y^2 - 2x}.$$

3.3.2. Find the equation of the tangent line of the graph of the equation $x^2y + xy^2 - 2x - y = 2$ at point $(1, 2)$.

Near point $(1, 2)$, this equation defines implicitly a function $y = f(x)$. The slope of the tangent line at this point is the derivative of this function. Use implicit differentiation to find the derivative:

$$2xy + x^2y' + y^2 + 2xyy' - 2 - y' = 0. \quad 4 + y' + 4 + 4y' - 2 - y' = 0. \quad 4y' + 6 = 0. \quad y' = -\frac{3}{2}.$$

The equation of the tangent line has the form $y = -\frac{3}{2}x + b$. $2 = -\frac{3}{2} + b$, $b = \frac{7}{2}$. The equation is $y = -\frac{3}{2}x + \frac{7}{2}$, or $3x + 2y = 7$.

3.3.3. Find the derivative of the implicit function $y = f(x)$, at the point at $(0, \pi)$, where y is defined by the equation $\sin(x + y) = \frac{2x}{y}$.

Taking the derivative with respect to x on both sides of the equation, we have

$$(\cos(x + y))(1 + y') = \frac{2(y - xy')}{y^2}.$$

When $x = 0$, and $y = \pi$, we have $\cos \pi(1 + y') = \frac{2}{\pi}$. Then $-1 - y' = \frac{2}{\pi}$. Solve this equation for y' , we have $y' = -\frac{2}{\pi + 1}$

3.3.4. Find the equation of the tangent line at $(1, 0)$ of a function $y = f(x)$ defined implicitly by the equation $xe^y + ye^x = 1$.

Taking the derivative with respect to x on both sides of the equation, we have

$e^y + xe^y y' + y'e^x + ye^x = 0$. When $x = 1$, and $y = 0$, $1 + y' + ey' = 0$. Then $(1 + e)y' = -1$, and $y' = \frac{1}{1 + e}$. The equation of the tangent line is $y = \frac{1}{1 + e}(1 - x)$, or $x + (1 + e)y = 1$.

3. Derivatives of Inverse Trigonometric Functions

(1) $y = \arcsin x$

The function $y = \arcsin x$ can be defined implicitly by the equation $\sin y = x$. Taking the derivative with respect to x on both sides, we have $y' \cos y = 1$. Then $y' = 1 / \cos y$. When $-\pi/2 \leq y \leq \pi/2$, $\cos y \geq 0$. Hence,

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \quad \text{and} \quad y' = \frac{1}{\sqrt{1 - x^2}}.$$

(2) $y = \arccos x$

In a similar way, write $\cos y = x$. Then $-y' \sin y = 1$, and $y' = -1 / \sin y$. When $0 \leq y \leq \pi$, $\sin y \geq 0$, $\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$. Hence $y' = -\frac{1}{\sqrt{1 - x^2}}$.

(3) $y = \arctan x$

Write $\tan y = x$. Then $\frac{y'}{\cos^2 y} = 1$ and $y' = \cos^2 y = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$.

Examples

3.3.6 $y = x \arctan \sqrt{x}$.

$$y' = \arctan \sqrt{x} + \frac{x}{2\sqrt{x}(1+x)} = \arctan \sqrt{x} + \frac{\sqrt{x}}{2(1+x)}.$$

3.3.7. $y = \arcsin x + \arccos x$.

$y' = \frac{1}{\sqrt{1-x^2}} + \left(-\frac{1}{\sqrt{1-x^2}}\right) = 0$. Hence, this function is a constant. Since $y(0) = \frac{\pi}{2}$, we have identity $\arcsin x + \arccos x = \frac{\pi}{2}$.

§ 3.6. DERIVATIVES OF LOGARITHMIC FUNCTIONS

1. The Derivative of Logarithmic Functions

Since the natural logarithmic function is the inverse of the natural exponential function, function $y = \ln x$ is defined implicitly by an equation $x = e^y$. Using implicit differentiation, we have $1 = y'e^y$. Hence, $y' = 1/e^y = 1/x$.

By the chain rule, we also have that the derivative of function $y = \ln(-x)$ is also $y' = 1/x$.

Use the base changing formula, $y = \log_a x = \ln x / \ln a$. Then $y' = 1/((\ln a)x)$.

Examples

3.4.1. $y = \ln(\sin x)$. $y' = \cos x / \sin x = \cot x$.

3.4.2. $y = \log_2(x^2 + 1)$. $y' = \frac{2x}{(\ln 2)(x^2 + 1)}$.

3.4.3. $y = x \ln x$. $y' = \ln x + 1$.

3.4.4. $y = \ln x / x$. $y' = \frac{\frac{1}{x}x - \ln x}{(\ln x)^2} = \frac{1 - \ln x}{(\ln x)^2}$.

3.4.5. $y = (\ln x)^2$, $y' = 2 \ln x / x$.

2. Logarithmic Differentiation

Logarithmic differentiation is use in two cases:

Case 1. The function is a complicated product and/or quotient. In this case the logarithm of this function is a sum or difference. The derivative is easier to calculate.

Examples

$$3.4.6. \quad y = x^{3/4}(x^2 + 1) / (3x + 2)^5.$$

$$\ln y = \ln x^{3/4} + \ln(x^2 + 1) - \ln(3x + 2)^5 = (3/4)\ln x + \ln(x^2 + 1) - 5 \ln(3x + 2).$$

Taking the derivative on both sides, we have

$$\frac{y'}{y} = \frac{3}{4x} + \frac{2x}{x^2 + 1} - \frac{15}{3x + 2},$$

$$y' = \left(\frac{x^{3/4}(x^2 + 1)}{(3x + 2)^5} \right) \left(\frac{3}{4x} + \frac{2x}{x^2 + 1} - \frac{15}{3x + 2} \right)$$

$$3.4.7. \quad y = \frac{e^{x^2+1}(2x-1)^3}{x^5}. \quad \text{Find } y'(1).$$

$$\ln y = (x^2 + 1) + 3 \ln(2x - 1) - 5 \ln x.$$

$$\frac{y'}{y} = 2x + \frac{6}{2x-1} - \frac{5}{x} = \frac{2x^2(2x-1) + 6x - 5(2x-1)}{x(2x-1)} = \frac{4x^3 - 2x^2 - 4x + 5}{x(2x-1)}.$$

$$\text{When } x = 1, \quad \frac{y'}{y} = 3. \quad y(1) = e^2. \quad y' = 3e^2.$$

Case 2. The function is of the form $y = f(x)^{g(x)}$.

Taking the logarithm on both sides, we have $\ln y = g(x)\ln f(x)$. The derivative with respect to x gives $\frac{y'}{y} = g(x)\ln f(x) + \frac{g(x)}{f'(x)}$. Hence,

$$y' = y \left(g(x)\ln f(x) + \frac{g(x)}{f'(x)} \right) = f(x)^{g(x)} \left(g(x)\ln f(x) + \frac{g(x)}{f'(x)} \right)$$

$$3.4.8. \quad y = x^{\sqrt{x}}.$$

$$\text{Then } \ln y = \sqrt{x} \ln x. \quad \frac{y'}{y} = \frac{1}{2\sqrt{x}} \ln x + \frac{\sqrt{x}}{x} = \frac{\ln x + 2}{2\sqrt{x}}. \quad y' = x^{\sqrt{x}} \left(\frac{\ln x + 2}{2\sqrt{x}} \right)$$

$$3.4.9. \quad y = (x + 1)^{\sin x}. \quad \text{Find } y'(0).$$

$\ln y = \sin x \ln(x + 1)$. $\frac{y'}{y} = \cos x \ln(x + 1) + \frac{\sin x}{x+1}$. When $x = 0$, $\frac{y'}{y} = 0$. Hence, $y'(0) = 0$.

§ 3.9. RELATED RATES

Type of the Questions: There are two (or three) functions of the same variable (usually time t). These functions are related by an equation. The rate of change (i.e., the derivative) of one function is unknown, and the rate of change of the other function(s) is known. We want to find the unknown rate of change (i.e., the derivative) at a given value of the variable.

Example 3.5.1. Suppose a right triangle has the length of the hypotenuse 50cm, and the length of the other two sides are x and y . If x is decreasing at a rate 4 cm / second, what is the rate of change of y when $x = 30$ cm?

Two functions: $x(t)$ and $y(t)$.

Relation between two functions:

$$x^2 + y^2 = 50^2.$$

The rate of change of x is -4 cm/second. (Note the negative sign!)

Question: Find the rate of change of y at time t when $x = 30$ cm.

Steps of solution:

1. Identify the question:

Identify two functions: One has a known rate of change, and the other function whose rate of change is to be found.

Find the equation that relates these two functions.

In this example, we have two functions $x(t)$ and $y(t)$. The derivative of $x(t)$ is $x'(t)$ is known, and the derivative $y'(t)$ is unknown. The relation between these two functions is $x^2 + y^2 = 50^2$.

2. Take the derivative of the equation with respect to the common variable (this common variable is usually the time t , not x).

In this example, taking the derivative of the relation with respect to t , we have $2xx' + 2yy' = 0$.

3. Express the unknown derivative as a function of the values of the functions and the known derivative(s).

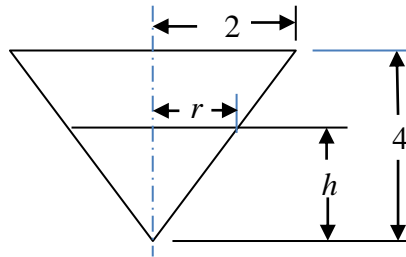
In this example, when $x = 30$, $y = 40$. Since $x' = -4$, $y' = -xx' / y = 3$.

Examples

3.5.2. Suppose oil spills from a tanker and spreads in a circular pattern. How fast is the area of the spill increasing when the radius is 30 m and it is increasing at a rate 1 m/s?

The related functions are the area $A(t)$ and the radius $r(t)$. The relation is $A = \pi r^2$. Taking the derivative with respect to t , we have $A' = 2\pi r r'$. Since $r = 30$ and $r' = 1$, $A' = 60\pi \approx 188.5 \text{ m}^2 / \text{s}$.

3.5.3. A water tank has the shape of an inverted circular cone. The radius of the top is 2 meters and the height of the tank is 4 meters. If water is pumped in at a rate $2 \text{ m}^3 / \text{min}$, find rate of water level increasing when the water is 3 m deep.



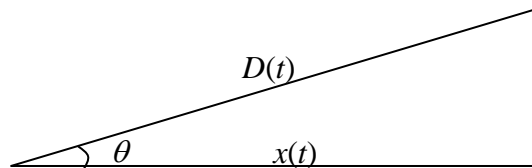
Functions of t : volume of water in the tank $V(t)$, height of water in the tank $h(t)$.

Relation: $V = \pi r^2 h / 3$. By similar triangles, $r = 2h / 4 = h / 2$. Hence, $V = \pi h^3 / 12$.

Take the derivative with respect to t : $V' = \pi h^2 h' / 4$.

Since $V' = 2$ and $h = 3$, we have $h' = 4V' / (\pi h^2) = 8 / (9\pi) \approx 0.28 \text{ m/min}$.

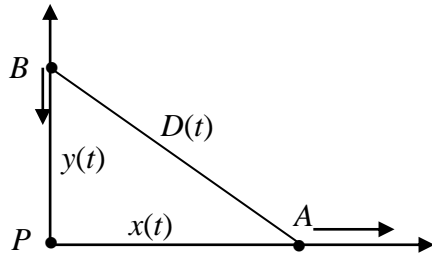
3.5.4. An airplane flies horizontally towards a radar station. At one moment, the distance between the radar and the airplane is 25 km and is decreasing at a rate 200 m/s. The angle of elevation is 0.2 radian and is increasing at a rate 0.004 radian / sec. Find the speed of the airplane?



Let the horizontal distance between the radar and the airplane be $x(t)$. Then the speed of the airplane is $x'(t)$. Let the angle of elevation be $\theta(t)$ and let the distance between the radar and the airplane be $D(t)$.

Then $x = D \cos \theta$, and $x' = D' \cos \theta - D\theta' \sin \theta = -200 \cos 0.2 - 25000 \times 0.004 \times \sin 0.2 \approx -215.88$ m / s. The speed is approximately 216 meters per second.

3.5.5. Car A is moving from a point P towards east, and car B is moving south towards point P. When car A is moving at a speed 25 m/sec at 40 km east of point P, it is found that the distance between A and B is 50 km, and this distance is increasing at a rate 1 m/sec. What is the speed of car B?



Let the distance between car A and point P be x , and let the distance between car B and point P be y .

Then the distance D between car A and car B is calculated by the equation $D^2 = x^2 + y^2$.

Taking the derivative with respect to time t : $2DD' = 2xx' + 2yy'$, or $DD' = xx' + yy'$.

When $D = 50$, and $x = 40$, $y = 30$. Since $x' = 25$ and $D' = -1$, we have

$$y' = \frac{DD' - xx'}{y} = \frac{50 \times (-1) - 40 \times 25}{30} = -\frac{1050}{30} = -35.$$

Car B is approaching point P at a speed 35 meters per second.

3.5.6. Suppose $F = f(t)$ and $G = g(t)$ are two functions related by the equation $F^2 + 4G = FG^2 - 2$, $F \geq 0$, $G \geq 0$. Find G' when $F = 2$ and $F' = -4$.

Solution. $2FF' + 4G' = F'G^2 + 2FGG'$. $G' = \frac{(2F - G^2)F'}{2FG - 4}$ When $F = 2$, $4 + 4G = 2G^2 - 2$,

$$G^2 - 2G - 3 = 0, G = 3, -1. \text{ Since } G \geq 0, \text{ take } G = 3. \text{ Hence, } G' = \frac{(4 - 9)(-4)}{12 - 4} = \frac{5}{2}.$$

§ 3.10. LINEAR APPROXIMATIONS AND DIFFERENTIALS

Consider function $y = f(x)$. The slope of the tangent line at a point (x_0, y_0) is $f'(x_0)$, and the equation of the tangent line is $L(x) = f'(x_0)(x - x_0) + y_0$, where $y_0 = f(x_0)$. Since the tangent line

at point (x_0, y_0) is close to the graph of this function near this point, we can use the value of the tangent line to be an approximation of the value of the function, i.e., when x is close to x_0 ,

$$f(x) \approx L(x) = f'(x_0)(x - x_0) + f(x_0).$$

Since the equation of the tangent line is linear, this is called the *linearization* or *linear approximation* of this function at this point.

Differentials: Let $dx = x - x_0$. Then $f(x_0 + dx) - f(x_0) \approx f'(x_0)dx = dy$.

The right-hand side $dy = df(x_0) = f'(x_0)dx$ is called the *differential* of $f(x)$. The differential can be used to approximate the change of the function:

$$\Delta y = f(x_0 + dx) - f(x_0) \approx dy = f'(x_0)dx.$$

Examples

3.6.1. Find the linearization of $y = \sin x$ at $x = 0$.

$y' = \cos x$. $x = 0$, $\cos 0 = 1$. $\sin 0 = 0$. The linearization is $y = 1 \times (x - 0) + 0 = x$.

This means that, when x is close to 0, $\sin x \approx x$.

3.6.2. Use the linearization of the function $y = (3x + 11)^{1/5}$ to find an estimate of $\sqrt[5]{30}$. Note that $y(7) = 2$.

$y' = \frac{3}{5(3x+11)^{4/5}}$. $y'(7) = \frac{3}{80}$. The linearization of this function at $x = 7$ is $y = \frac{3}{80}(x - 7) + 2$.

Using this function, let $\sqrt[5]{30} = \sqrt[5]{3x+11}$. Then $3x + 11 = 30$, $x = \frac{19}{3}$. By the linear

approximation, $\sqrt[5]{30} \approx \frac{3}{80} \left(\frac{19}{3} - 7 \right) + 2 \approx 1.975$.

3.6.3. If the measurement of the radius of a sphere is r with an error $\pm x$, what are the possible absolute error and the relative error of the volume of the sphere?

The volume of a sphere is $V = \frac{4}{3} \pi R^3$, where R is the radius of a sphere. The differential of V is $dV = 4\pi R^2 dR$. (Note that $4\pi R^2$ is the area of the surface, and the differential is an approximate value of the volume of a layer near the surface). If $|dR| \leq x$, then, when $R = r$, the absolute error is $|dV| \leq 4\pi r^2 x$.

The relative error is $\left| \frac{dV}{V} \right| \leq \frac{4\pi r^2 x}{(4/3)\pi r^3} = \frac{3x}{r}$. Since x/r is the relative error of r , the relative error of V is three times the relative error of r .

§ 4.9. ANTIDERIVATIVES

1. Antiderivatives of a Function and the Indefinite Integral

If $F'(x) = f(x)$, then $F(x)$ is an *antiderivative* of $f(x)$. The antiderivative of a function $f(x)$ is not unique. For example, x^2 is an antiderivative of $2x$. $x^2 + 1$ is also an antiderivative of $2x$. In general, $x^2 + C$ is an antiderivative of $2x$, where C is an arbitrary constant. On the other hand, if $F'(x) = 2x$, then $F'(x) - (x^2)' = (F(x) - x^2)' = 0$, and $F(x) - x^2 = C$. In other words, every antiderivative of $2x$ can be expressed as $x^2 + C$ with a certain constant C .

In general, the antiderivative of a function $f(x)$ is a family of functions of the form $F(x) + C$, where $F(x)$ is a particular antiderivative of $f(x)$.

The family of all antiderivatives of a function $f(x)$ is denoted by $\int f(x)dx$, called the *indefinite integral* of $f(x)$. Function $f(x)$ is called the *integrand*, and x is called the *variable of integration*.

2. Properties of indefinite integrals

$$\int c dx = cx + C,$$

$$\int cf(x)dx = c \int f(x)dx,$$

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx.$$

Note that an indefinite integral is a family of functions. An equality involving indefinite integral means that every function on one side is a member of the family on the other side. In particular,

$$\int f(x)dx - \int f(x)dx = \int (f(x) - f(x))dx = \int 0dx = C \text{ is an arbitrary constant but not } 0.$$

3. Antiderivative formula in the form of indefinite integrals

$f(x)$	$\int f(x)dx$
$x^n (n \neq -1)$	$\frac{1}{n+1} x^{n+1} + C$
$1/x$	$\ln x + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\frac{1}{1+x^2}$	$\arctan x + C$

Examples

3.7.1. Find the antiderivatives of the function $f(x) = 4 \sin x + \frac{2x^3 - \sqrt{x}}{x} + \frac{20}{1+x^2}$.

$$\int \left(4 \sin x + \frac{2x^3 - \sqrt{x}}{x} + \frac{20}{1+x^2} \right) dx = \int \left(4 \sin x + 2x^2 - x^{-1/2} + \frac{20}{1+x^2} \right) dx$$

$$= -4 \cos x + \frac{2}{3} x^3 - 2\sqrt{x} + 20 \arctan x + C.$$

3.7.2. $F(x)$ is an antiderivative of $f(x) = x + \frac{2}{1+x^2}$, and $F(1) = \frac{\pi}{2}$. Find $F(x)$.

By the antiderivative formula, $F(x) = \frac{x^2}{2} + 2 \arctan x + C$.

Since $F(1) = \frac{1}{2} + \frac{\pi}{2} + C = \frac{\pi}{2}$, we have $C = -\frac{1}{2}$. Then $F(x) = \frac{x^2}{2} + 2 \arctan x - \frac{1}{2}$.

3.7.3. Suppose $F(x)$ is an antiderivative of $f(x) = \frac{(x-1)^2}{x}$, and $F(1) = \frac{1}{2}$. Find $F(2)$.

$$F(x) = \int \frac{(x-1)^2}{x} dx = \int \frac{x^2 - 2x + 1}{x} dx = \int \left(x - 2 + \frac{1}{x} \right) dx = \frac{1}{2}x^2 - 2x + \ln|x| + C.$$

Since $F(1) = \frac{1}{2} - 2 + C = \frac{1}{2}$, $C = 2$. Hence, $F(x) = \frac{x^2}{2} - 2x + \ln|x| + 2$, and

$$F(2) = 2 - 4 + \ln 2 + 2 = \ln 2.$$

3.7.4 If $y'' = \frac{x+1}{\sqrt{x}}$, $y'(1) = 0$, and $y(1) = 1$, find $y(x)$.

Write $y = x^{1/2} + x^{-1/2}$. Then $y' = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$. Since $y'(1) = \frac{2}{3} + 2 + C = 0$, $C = -\frac{8}{3}$.

$y = \frac{4}{15}x^{5/2} + \frac{4}{3}x^{3/2} - \frac{8}{3}x + C$. By the initial condition, $1 = -\frac{16}{15} + C$, $C = \frac{31}{15}$. Hence,

$$y = \frac{4}{15}x^{5/2} + \frac{4}{3}x^{3/2} - \frac{8}{3}x + 31/15.$$

Application. From the acceleration function to the position function

Suppose an object is moving with a constant acceleration a . Since the acceleration is the second derivative of the position function $s = s(t)$, we have $s'' = a$. Then $s' = at + C$. When $t = 0$, $s'(0) = C$, which is the initial velocity v_0 . $s' = at + v_0$. Taking the antiderivative again, we have $s = \frac{1}{2}at^2 + v_0t + C_1$. When $t = 0$, $s(0) = C_1$, which is the initial position of the object s_0 . Finally, we have $s = \frac{1}{2}at^2 + v_0t + s_0$.

For a freely falling object, $a = -g$, where g is the acceleration of gravity. In this case, the height of the object is $h = \frac{1}{2}gt^2 + v_0t + h_0$.

Example 3.7.5. Suppose $h_0 = 40$ m, $v_0 = 10$ m/sec, $g = -10$ m/sec².

Then $v = 10 - 10t$, and $h = 40 + 10t - 5t^2$, $0 \leq t < \infty$.

When $v = 0$, the object reaches the maximum height. Let $v = 10 - 10t = 0$. Then $t = 1$. The maximum height is reached at $t = 1$ second and the maximum height is $h(1) = 45$ meters.

When $h = 40 + 10t - 5t^2 = 0$, this object hits the ground. Then $t^2 - 2t - 8 = 0$, and $t = 4$, $t = -2$. After 4 seconds the object will hit the ground.

From the properties of $f(x)$ to study the properties of its antiderivatives.

$f(x)$	$F(x)$
+	inc.
-	dec.
inc	concave up
dec	concave down

Example 3.7.6. Suppose the graph of $F'(x) = f(x)$ looks like the following. If $F(0) = 0$, sketch the graph of function $y = F(x)$.

