# Solution to Test 2 (version C)

MAT1320B, Fall 2019
Total = 20 marks

**AECDDB**

**Part I. Multiple-Choice Questions (2 × 6 = 12 marks)**

1. Some values of a function $y = f(x)$ are given as follows:

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 3 | 3.5 | 4 | 4.5 | 5 |
| $f(x)$ | 2 | 3 | 5 | 8 | 12 |

A Riemann sum of $f(x)$ on interval [3, 5] with $n = 4$ and equally spaced mesh points taking the left end as the sample point in each subinterval equals

(A) 9; (B) 15; (C) 18; (D) 20; (E) 14; (F) 8.

*Solution*. (A) The step size is $h = 0.5$. The Riemann sum is

$S(4) = 0.5 \times 2 + 0.5 \times 3 + 0.5 \times 5 + 0.5 \times 8 = 0.5 \times (2 + 3 + 5 + 8) = 9.$

2. The derivative of the function $f(x) = (\sin x)^x$ at $x = \dfrac{\pi}{2}$ is

(A) 4; (B) 5; (C) 2; (D) 3; (E) 0; (F) 1.

*Solution*. (E) Take the logarithm of the function: $\ln f(x) = x \ln(\sin x)$.

$f'(x) = f(x)\left(\ln(\sin x) + x\dfrac{\cos x}{\sin x}\right) = (\sin x)^x\left(\ln(\sin x) + x\dfrac{\cos x}{\sin x}\right)$ When $x = \dfrac{\pi}{2}$, $f'(1) = 0$.

3. The derivative of the function $F(x) = \int_1^{x^2} t^2\,dt$ is

(A) $2x^4$; (B) $2x^3$; (C) $2x^5$; (D) $x^5$; (E) $x^4$; (F) $x^3$.

*Solution*. (C) Let $u = x^2$. Then $y = F(x) = \int_1^u t^2\,dt$. $y_u' = u^2 = x^4$ and $u_x' = 2x$. By the chain rule, $y_x' = y_u' u_x' = 2x^5$.

4. If $f''(x) = \dfrac{2 - 3\sqrt{x}}{x^3}$, $f'(1) = 1$, $f(1) = -1$, then $f(4) =$

(A) 1; (B) $\frac{2}{3}$; (C) $\frac{1}{2}$; (D) $\frac{1}{4}$; (E) $\frac{1}{3}$; (F) 0.

*Solution*. (D) $f'(x) = \int \frac{2-3\sqrt{x}}{x^3}dx = \int (2x^{-3} - 3x^{-5/2})dx = -x^{-2} + 2x^{-3/2} + C$.

$f'(1) = 1 + C = 1$. Then $C = 0$, and $f'(x) = -x^{-2} + 2x^{-3/2}$.

$f(x) = \int \left(-x^{-2} + 2x^{-3/2}\right)dx = x^{-1} - 4x^{-1/2} + C$

$f(1) = -3 + C = -1$. Then $C = 2$, and $f(x) = x^{-1} - 4x^{-1/2} + 2$. $f(4) = \frac{1}{4}$.

5. The approximation of $\sqrt{15}$ obtained by the linearization of the function $f(x) = \sqrt{5x+1}$ at $x = 3$ is

(A) $\frac{19}{5}$; (B) $\frac{23}{6}$; (C) $\frac{27}{7}$; (D) $\frac{31}{8}$; (E) $\frac{34}{9}$; (F) $\frac{35}{9}$.

*Solution*. (D) $f'(x) = \frac{5}{2\sqrt{5x+1}}$. $f'(3) = \frac{5}{8}$. Since $f(3) = 4$, the linearization of $f(x)$ at $x = 3$ is $L(x) = \frac{5}{8}(x-3) + 4$. To find an approximation of $\sqrt{15}$, let $15 = 5x + 1$. Then $x = \frac{14}{5}$. Hence,

$\sqrt{15} \approx L\left(\frac{14}{5}\right) = \frac{5}{8}\left(\frac{14}{5} - 3\right) + 4 = -\frac{1}{8} + 4 = \frac{31}{8}$.

6. $\int_0^1 \arctan x dx =$

(A) $\frac{1}{4}\pi + \ln 2$; (B) $\frac{1}{4}(\pi - 2\ln 2)$; (C) $\frac{1}{4}(\pi + \ln 2)$;
(D) $\frac{1}{2}(\pi - \ln 2)$; (E) $\frac{1}{2}(\pi - 2\ln 2)$; (F) $\frac{1}{2}\pi - \ln 2$.

*Solution*. (B) Use integration by parts. Let $u = \arctan x$, and let $v' = 1$. Then $u' = \frac{1}{1+x^2}$, and $v = x$.

$\int_0^1 \arctan x = [\arctan x]_{x=0}^{1} - \int_0^1 \frac{x}{1+x^2}dx$.

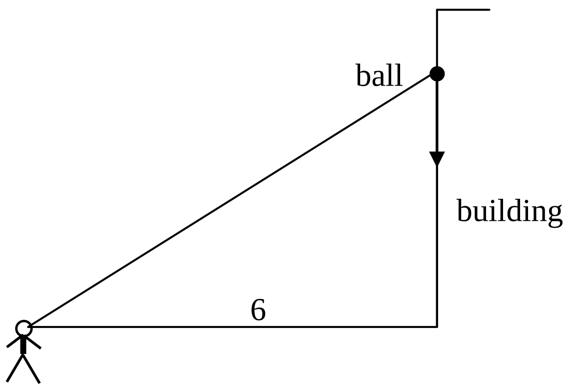
To find the integral on the right-hand side, use variable substitution. Let $u = 1 + x^2$. Then $u' = 2x$.

$$\int_0^1 \frac{x}{1+x^2}\,dx = \int_1^2 \frac{x}{1+x^2}\left(\frac{1}{2x}\right)du = \frac{1}{2}\int_1^2 \frac{1}{u}\,du = \frac{1}{2}[\ln|u|]_{u=1}^2 = \frac{\ln 2}{2}.$$

Hence, $\int_0^1 \arctan x = [\arctan x]_{x=0}^1 - \int_0^1 \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{\ln 2}{2} = \frac{1}{4}(\pi - 2\ln 2)$.

**Part II. Detailed Answer Question (8 marks)**

1. (4 marks) A ball is falling down from the top of a building. A person is 6 meters away from the building. When the distance between the ball and the person's eyes is 10 meters, this distance is reducing at a rate 2.4 m/sec. What is the rate of change of the height of the ball at this time?




*Solution*. Let $h(t)$ be the height of the ball above the eyes of the person, and let $D(t)$ be the distance between the object and the person's eyes. Then $h^2 + 6^2 = D^2$. Take the derivative with respect to $t$. $2hh' = 2DD'$. $h' = \frac{DD'}{h}$. When $D = 10$ m, $h = \sqrt{10^2 - 6^2} = 8$ m. Hence,

The rate of change of the height of the ball at this time is $h' = \frac{10\times(-2.4)}{8} = -3$ m/sec.

2. (4 marks) Evaluate $\int_0^1 \frac{1}{\sqrt{x}+1}\,dx$.

*Solution*. Use variable substitution. Let $u = \sqrt{x} + 1$. Then $u' = \frac{1}{2\sqrt{x}}$. When $x = 0$, $u = 1$, and, when $x = 1$, $u = 2$.

$$\int_0^1 \frac{1}{\sqrt{x}+1}\,dx = \int_1^2 \frac{2\sqrt{x}}{\sqrt{x}+1}\,du = 2\int_1^2 \frac{u-1}{u}\,du = 2[u - \ln u]_{u=1}^2$$
$= 2((2 - \ln 2) - (1 - \ln 1)) = 2(1 - \ln 2)$.