Name: ________________________ Mark:__________/20

Student ID: ________________________

UNIVERSITY OF OTTAWA
DEPARTMENT OF MATHEMATICS & STATISTICS

MAT 2342 (Fall 2019)

Midterm 2 (Thursday, November 21, 2019)

Time: 80 minutes (no cellphones, notes, books, talking).

MARKS

(5) 1. Find the projection of the vector $\vec{v} = (1, 1, 1)$ on the column space of the following matrix. (Column space means the space generated by the columns of the matrix)

$$\begin{array}{cc} v_1 & v_2 \end{array}$$
$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & -1 \end{bmatrix}$$

1 mark

$$v_1 \cdot v_2 = (1, 2, -1) \cdot (1, -1, -1) = 0 \Rightarrow \boxed{v_1 \perp v_2}$$

$$\boxed{\operatorname{proj}_A v = \operatorname{proj}_{v_1} v + \operatorname{proj}_{v_2} v} = \frac{v \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{v \cdot v_2}{v_2 \cdot v_2} v_2$$

2

$$= \boxed{\frac{2}{6} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} + \boxed{\frac{-1}{3} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}} = \begin{pmatrix} 1/3 \\ 2/3 \\ -1/3 \end{pmatrix} + \begin{pmatrix} -1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

1 1

(4) 2. If U is the space generated by vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$, find a basis for $U^{\perp}$.

1 mark

$$\boxed{\left[\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]} \xrightarrow{\substack{R_1 \\ R_2 - R_1 \\ R_3 + R_1 \\ R_4}} \boxed{\left[\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]}$$

2

$$\Rightarrow \boxed{U^{\perp} = \left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle}$$

1

(5) 3. Find the best linear approximation for the following given set of points

x	-2	-1	0	1	2
y	1	0	1	2	3

$\alpha x + \beta = y$

$-2\alpha + \beta = 1$

$-\alpha + \beta = 0$

$\beta = 1$

$\alpha + \beta = 2$

$2\alpha + \beta = 3$

$$\boxed{A = \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}}$$ 1 mark

$$\boxed{A^t A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = A^t b}$$ 1

$$\Rightarrow \begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}}$$ 2

$$\Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1/10 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \boxed{\begin{bmatrix} 6/10 \\ 7/5 \end{bmatrix}}$$ 1

$\Rightarrow y = \frac{6}{10}x + \frac{7}{5}$ is the best line approximation.

4. Consider the quadratic form $Q = 2x_1^2 + 8x_1x_2 + 2x_2^2$. If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then

(1) (a) find the symmetric matrix A s.th $Q = x^t A x$,

(3) (b) diagonalize A,

(2) (c) and find a suitable change of variables to y_1 and y_2 s.th $Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$ for $\lambda_{1,2}$ eigenvalues of A.

a) $A = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 12$

$= (\lambda - 6)(\lambda + 2)$

b) $(A - 6I)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}$ 1

$(A + 2I)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}}$ 1

$\boxed{P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \quad D = \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix}}$ 1

$A = PDP^t$

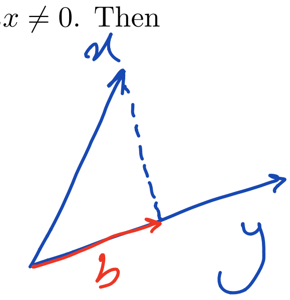
$\boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P^t x =} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boxed{\begin{pmatrix} 1/\sqrt{2}\,(x_1 + x_2) \\ 1/\sqrt{2}\,(x_1 - x_2) \end{pmatrix}}$ 1

1

Bonus

5. Circle the correct answer. No need to show the work.

(1) (a) Let x and y be two vectors and let $b = Proj_y x \neq 0$. Then



i. x and y are perpendicular.
ii. x and b are perpendicular.
iii. b and y are perpendicular.
iv. Non of the above.

(1) (b) Let $A = [v_1, v_2, v_3]$ be a 3×3 matrix, where the column vectors $v_1, v_2,$ and v_3 give an orthonormal basis for $\mathbb{R}^3$. Then

i. $A = A^{-1}$ ii. $A = A^t$ iii. $A^t = A^{-1}$ iv. $A = I$

(1) (c) Let $Q = x^t A x$ be a quadratic form given by a symmetric matrix A. For which A the quadratic form is positive definite?

i. $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ ii. $A = \begin{bmatrix} 0 & 1 \\ 1 & \frac{3}{2} \end{bmatrix}$ iii. $A = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$ iv. $A = \begin{bmatrix} 1 & 1 \\ 1 & \frac{5}{2} \end{bmatrix}$

positive definite $\Leftrightarrow$ all eigenvalues are $+$.

i : $\lambda_1, \lambda_2 = 2, -2$

ii : $\lambda_1, \lambda_2 = -\frac{1}{2}, 2$

iii : $\lambda_1, \lambda_2 = 1, -3$

iv : $\lambda_1, \lambda_2 = \frac{1}{2}, 3$