

MAT2122 Multivariable Calculus (Fall 2018)

Final solutions (70 points)

1 (7 points). Find an equation of the tangent plane to the graph of the function $f(x, y) = (x - y)e^{x+y}$ at the point $(1, -1)$. What is the distance from the origin $(0, 0, 0)$ to this plane?

Solution: The equation of the tangent plane to the graph of a function f at the point (x_0, y_0) is

$$z = f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0) . \checkmark$$

In our case

$$\begin{aligned} f'_x &= e^{x+y} + (x - y)e^{x+y} = (x - y + 1)e^{x+y} , \checkmark \\ f'_y &= -e^{x+y} + (x - y)e^{x+y} = (x - y - 1)e^{x+y} , \end{aligned}$$

whence

$$f'_x(1, -1) = 3 , \quad f'_y(1, -1) = 1 . \checkmark$$

Since $f(1, -1) = 2$, \checkmark the equation of the tangent plane is

$$z = 2 + 3(x - 1) + (y + 1) ,$$

or

$$3x + y - z = 0 . \checkmark$$

The point $(0, 0, 0)$ satisfies this equation, and therefore belongs to the plane, so that the distance between the point $(0, 0, 0)$ and the tangent plane is 0. \checkmark

Absence of minor mistakes. \checkmark

2 (6 points). Let $f(x, y) = (\cos y + x^2, e^{x+y})$ and $g(u, v) = (e^{u^2}, u - \sin v)$. Write a formula for $f \circ g$. Calculate the derivative matrix $\mathbf{D}(f \circ g)(0, 0)$ using the chain rule.

Solution:

$$\begin{aligned} f \circ g(u, v) &= f(g(u, v)) = f(e^{u^2}, u - \sin v) = \left(\cos(u - \sin v) + (e^{u^2})^2, e^{(e^{u^2} + u - \sin v)} \right) \\ &= \left(\cos(u - \sin v) + e^{2u^2}, e^{(e^{u^2} + u - \sin v)} \right) . \checkmark \end{aligned}$$

The derivative matrices of the functions f and g are, respectively,

$$\mathbf{D}f(x, y) = \begin{pmatrix} 2x & -\sin y \\ e^{x+y} & e^{x+y} \end{pmatrix} , \checkmark$$

and

$$\mathbf{D}g(u, v) = \begin{pmatrix} 2ue^{u^2} & 0 \\ 1 & -\cos v \end{pmatrix} , \checkmark$$

whence

$$\mathbf{D}g(0, 0) = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} ,$$

and, since $g(0, 0) = (1, 0)$, \checkmark

$$\mathbf{D}f(g(0, 0)) = \mathbf{D}f(1, 0) = \begin{pmatrix} 2 & 0 \\ e & e \end{pmatrix} .$$

Then by the chain rule

$$\mathbf{D}(f \circ g)(0, 0) = \mathbf{D}(f)(g(0, 0))\mathbf{D}g(0, 0) = \begin{pmatrix} 2 & 0 \\ e & e \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e & -e \end{pmatrix}. \quad \checkmark$$

Absence of minor mistakes. \checkmark

3 (7 points). Find and classify the critical points of the function $f(x, y) = x^3 + x^2y - y^2 - 4y$.

Solution:

$$\nabla f = (f'_x, f'_y) = (3x^2 + 2xy, x^2 - 2y - 4), \quad \checkmark$$

so that the critical points of f are the solutions of the system

$$\begin{cases} 3x^2 + 2xy = 0 \\ x^2 - 2y - 4 = 0 \end{cases} \iff \begin{cases} x(3x + 2y) = 0 \\ x^2 - 2y - 4 = 0 \end{cases}. \quad \checkmark$$

From the first equation either $x = 0$ or $-2y = 3x$. In the first case the second equation yields $y = -2$, whereas in the second case $x^2 + 3x - 4 = 0$, whence either $x = 1, y = -3/2$, or $x = -4, y = 6$. Thus, the function f has 3 critical points

$$P_1 = (0, -2), \quad P_2 = (1, -3/2), \quad P_3 = (-4, 6). \quad \checkmark$$

Since

$$f''_{xx} = 6x + 2y, \quad f''_{yy} = -2, \quad f''_{xy} = 2x, \quad \checkmark$$

we have

$$Hf(x, y) = \begin{pmatrix} 6x + 2y & 2x \\ 2x & -2 \end{pmatrix}, \quad \checkmark$$

so that

$$Hf(P_1) = \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}$$

has positive determinant and negative diagonal entries, whence P_1 is a local maximum;

$$Hf(P_2) = \begin{pmatrix} 3 & 2 \\ 2 & -2 \end{pmatrix}$$

has negative determinant, whence P_2 is a saddle point;

$$Hf(P_3) = \begin{pmatrix} -12 & -8 \\ -8 & -2 \end{pmatrix}$$

has negative determinant, whence P_3 is a saddle point. \checkmark

Absence of minor mistakes. \checkmark

4 (6 points). By using Lagrange multipliers find the absolute minimum and maximum of the function $f(x, y) = x^2 - y^2$ on the region D defined by $x^2 + y^2 \leq 1$.

Solution: The gradient of f is $\nabla f = (2x, -2y)$, whence f has the only critical point $P_0 = (0, 0)$ which happens to lie inside D . \checkmark For finding boundary critical points one has to solve the system of equations

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = C \end{cases},$$

where $g(x, y) = x^2 + y^2$ and $C = 1$, or, equivalently,

$$\begin{cases} (2x, -2y) = \lambda(2x, 2y) \\ x^2 + y^2 = 1 \end{cases} \iff \begin{cases} x = \lambda x \\ y = -\lambda y \\ x^2 + y^2 = 1 \end{cases} \quad \checkmark$$

From the first equation either $x = 0$, or $\lambda = 1$. In the first case from the third equation $y = \pm 1$, and then from the second equation $\lambda = -1$, which gives critical points $P_1 = (0, -1), P_2 = (0, 1)$. In the second case from the second equation $y = 0$, whence from the third equation $x = \pm 1$, which gives critical points $P_3 = (-1, 0), P_4 = (1, 0)$. \checkmark The values of f at the critical points are

$$f(P_0) = 0, \quad f(P_1) = f(P_2) = -1, \quad f(P_3) = f(P_4) = 1. \quad \checkmark$$

Therefore, the global maximum of f is 1 attained at the points $P_3 = (-1, 0), P_4 = (1, 0)$, and the global minimum of f is -1 attained at the points $P_1 = (0, -1), P_2 = (0, 1)$. \checkmark

Absence of minor mistakes. \checkmark

5 (7 points). Find the volume of the solid W bounded by $2x^2 + y^2 = 2, z = 0$, and $x + y + 2z = 2$.

Solution: The body W is determined by the footprint

$$D = \{(x, y) : 2x^2 + y^2 \leq 2\} \quad \checkmark$$

and the roof function

$$z = \frac{2 - x - y}{2} = 1 - \frac{x}{2} - \frac{y}{2}. \quad \checkmark$$

In order to find

$$\text{vol } W = \iint_D \left(1 - \frac{x}{2} - \frac{y}{2}\right) dx dy \quad \checkmark$$

change variables as

$$\begin{cases} x = r \cos \phi \\ y = \sqrt{2}r \sin \phi \end{cases}$$

with $0 \leq r \leq 1, 0 \leq \phi \leq 2\pi$. \checkmark Then

$$\frac{\partial(x, y)}{\partial(r, \phi)} = \begin{vmatrix} \cos \phi & -r \sin \phi \\ \sqrt{2} \sin \phi & \sqrt{2}r \cos \phi \end{vmatrix} = \sqrt{2}r, \quad \checkmark$$

whence

$$\begin{aligned} \text{vol } W &= \int_0^{2\pi} \int_0^1 \left(1 - \frac{1}{2}r \cos \phi - \frac{\sqrt{2}}{2}r \sin \phi\right) \sqrt{2}r dr d\phi \\ &= \sqrt{2} \int_0^{2\pi} \int_0^1 r dr d\phi - \frac{\sqrt{2}}{2} \int_0^{2\pi} \int_0^1 r^2 \cos \phi dr d\phi - \int_0^{2\pi} \int_0^1 r^2 \sin \phi dr d\phi \\ &= \sqrt{2} \cdot 2\pi \cdot \frac{1}{2} = \sqrt{2}\pi. \quad \checkmark \end{aligned}$$

Absence of minor mistakes. \checkmark

6 (8 points). Find the average value of the function $f(x, y) = x^2y$ on the planar domain bounded by the graphs of the functions $y = x^2, y = 2x - 1, y = -2x - 1$.

Solution: The domain D is bounded by the graphs of 3 functions, whose pairwise intersections are the points obtained by solving the equations

$$\begin{aligned}x^2 &= 2x - 1 &\implies x = 1 &\implies y = x^2 = 2x - 1 = 1, \\x^2 &= -2x - 1 &\implies x = -1 &\implies y = x^2 = -2x - 1 = 1, \\2x - 1 &= -2x - 1 &\implies x = 0 &\implies y = 2x - 1 = -2x - 1 = -1.\end{aligned}$$

Therefore, the projection of the domain D onto the horizontal line is the interval $[-1, 1]$, and D can be described as

$$D = \{(x, y) : -1 \leq x \leq 1, \phi(x) \leq y \leq x^2\},$$

where the function ϕ is defined as

$$\phi(x) = \begin{cases} -2x - 1, & -1 \leq x \leq 0, \\ 2x - 1, & 0 \leq x \leq 1. \end{cases}$$

The average value of f is

$$\bar{f} = \frac{I}{\text{area } D},$$

where (because of the symmetry of the domain D and of the function x with respect to the reflection $x \mapsto -x$)

$$\begin{aligned}I &= \iint_D f(x, y) \, dx dy = 2 \int_0^1 \int_{2x-1}^{x^2} x^2 y \, dy dx = \int_0^1 x^2 (x^4 - (2x-1)^2) \, dx \\ &= \int_0^1 (x^6 - 4x^4 + 4x^3 - x^2) \, dx = \frac{1}{7} - \frac{4}{5} + 1 - \frac{1}{3} = \frac{1}{105},\end{aligned}$$

and

$$\text{area } D = \int_{-1}^1 (x^2 - \phi(x)) \, dx = 2 \int_0^1 (x^2 - 2x + 1) \, dx = 2 \int_0^1 t^2 \, dt = \frac{2}{3},$$

whence

$$\bar{f} = \frac{\frac{1}{105}}{\frac{2}{3}} = \frac{1}{70}.$$

Absence of minor mistakes.

7 (9 points). Find the center of mass of the solid “bowl” W of constant density determined by the conditions $1 \leq x^2 + y^2 + z^2 \leq 4$ and $z \geq 0$.

Solution: Let the coordinates of the center of mass be (x_0, y_0, z_0) . Then $x_0 = y_0 = 0$ by rotational symmetry of W , whereas

$$z_0 = \frac{\iiint_W z \, dV}{\text{vol } W},$$

where

$$\text{vol } W = \frac{1}{2} \cdot \frac{4}{3} \pi (2^3 - 1^3) = \frac{14}{3} \pi.$$

As for the numerator, in spherical coordinates ✓

$$\begin{aligned} \iiint_W z \, dV &= \int_1^2 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \rho \cos \phi \cdot \rho^2 \sin \phi \, d\phi \quad \checkmark \\ &= \left[\int_1^2 \rho^3 \, d\rho \right] \cdot \left[\int_0^{2\pi} d\theta \right] \cdot \left[\int_0^{\pi/2} \cos \phi \sin \phi \, d\phi \right] \quad \checkmark \\ &= \frac{15}{4} \cdot 2\pi \cdot \frac{1}{2} = \frac{15}{4}\pi, \quad \checkmark \end{aligned}$$

whence

$$z_0 = \frac{\frac{15}{4}\pi}{\frac{14}{3}\pi} = \frac{45}{56}. \quad \checkmark$$

Absence of minor mistakes. ✓

8 (8 points). Verify Gauss' formula for the cube $W = [0, 1] \times [0, 1] \times [0, 1]$ and the vector field $\mathbf{F}(x, y, z) = (x^2y^2, xyz, z^3)$.

Solution: The divergence of the vector field \mathbf{F} is

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \checkmark \\ &= 2xy^2 + xz + 3z^2. \quad \checkmark \end{aligned}$$

By Gauss' formula

$$I = \iiint_W \operatorname{div} \mathbf{F} \, dV = \iint_{\partial W} \mathbf{F} \cdot d\boldsymbol{\xi}, \quad \checkmark$$

whence

$$I = \iiint_W (2xy^2 + xz + 3z^2) \, dV = 2 \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{4} + 1 = \frac{19}{12}. \quad \checkmark$$

In order to find I as a surface integral notice that ∂W consists of 6 square faces

$$\begin{aligned} S_1 &= \{(x, y, z) : x, y \in [0, 1], z = 0\}, \\ S_2 &= \{(x, y, z) : x, y \in [0, 1], z = 1\}, \\ S_3 &= \{(x, y, z) : x, z \in [0, 1], y = 0\}, \quad \checkmark \\ S_4 &= \{(x, y, z) : x, z \in [0, 1], y = 1\}, \\ S_5 &= \{(x, y, z) : y, z \in [0, 1], x = 0\}, \\ S_6 &= \{(x, y, z) : y, z \in [0, 1], x = 1\} \end{aligned}$$

with the outward normals \mathbf{n}_i equal to the vectors $-\mathbf{k}, \mathbf{k}, -\mathbf{j}, \mathbf{j}, -\mathbf{i}, \mathbf{i}$, respectively. Therefore,

$$\iint_{\partial W} \mathbf{F} \cdot d\boldsymbol{\xi} = \sum_{i=1}^6 \iint_{S_i} \mathbf{F} \cdot \mathbf{n}_i \, dS \quad \checkmark$$

with

$$\begin{aligned}
 \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1 dS &= - \int_0^1 \int_0^1 z^3 dx dy \Big|_{z=0} = 0, \\
 \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 dS &= \int_0^1 \int_0^1 z^3 dx dy \Big|_{z=1} = 1, \\
 \iint_{S_3} \mathbf{F} \cdot \mathbf{n}_3 dS &= - \int_0^1 \int_0^1 xyz dx dz \Big|_{y=0} = 0, \\
 \iint_{S_4} \mathbf{F} \cdot \mathbf{n}_4 dS &= \int_0^1 \int_0^1 xyz dx dz \Big|_{y=1} = \frac{1}{4}, \\
 \iint_{S_5} \mathbf{F} \cdot \mathbf{n}_5 dS &= - \int_0^1 \int_0^1 x^2 y^2 dy dz \Big|_{x=0} = 0, \\
 \iint_{S_6} \mathbf{F} \cdot \mathbf{n}_6 dS &= \int_0^1 \int_0^1 x^2 y^2 dy dz \Big|_{x=1} = \frac{1}{3},
 \end{aligned}$$

whence again

$$I = 1 + \frac{1}{4} + \frac{1}{3} = \frac{19}{12}.$$

Absence of minor mistakes. ✓

9 (6 points). Use Green's formula to find the area of the planar region D bounded by the curve $\mathbf{c}(t) = (\sin 2t, \sin t)$, $0 \leq t \leq \pi$.

Solution: The parametrization of \mathbf{c} is counterclockwise, ✓ so that

$$\begin{aligned}
 \text{area } D &= \frac{1}{2} \int_{\mathbf{c}} (-y dx + x dy) \quad \checkmark \\
 &= \frac{1}{2} \int_0^\pi (-y(t), x(t)) \cdot \mathbf{c}'(t) dt = \frac{1}{2} \int_0^\pi (-\sin t, \sin 2t) \cdot (2 \cos 2t, \cos t) dt \quad \checkmark \\
 &= \int_0^\pi (-\sin t \cos 2t + \sin t \cos^2 t) dt = \int_0^\pi (\cos^2 t - \cos 2t) \sin t dt = \int_0^\pi (1 - \cos^2 t) \sin t dt \quad \checkmark \\
 &= \left[\frac{1}{3} \cos^3 t - \cos t \right] \Big|_{t=0}^{t=\pi} = \frac{4}{3}. \quad \checkmark
 \end{aligned}$$

Absence of minor mistakes. ✓

10 (6 points). Determine whether the vector field $\mathbf{F}(x, y, z) = (2xyz, x^2z, x^2y)$ is conservative. If yes, then find an appropriate potential.

Solution: A vector field is conservative if and only if its curl vanishes. Since

$$\begin{aligned}
 \text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \quad \checkmark \\
 &= (x^2 - x^2, 2xy - 2xy, 2xz - 2xz) = 0, \quad \checkmark
 \end{aligned}$$

the field \mathbf{F} is indeed conservative. Its potential f must satisfy the relation $\nabla f = \mathbf{F}$, i.e.,

$$\begin{aligned}f'_x &= 2xyz, \\f'_y &= x^2z, \quad \checkmark \\f'_z &= x^2y,\end{aligned}$$

whence

$$\begin{aligned}f(x, y, z) &= x^2yz + h_1(y, z), \\f(x, y, z) &= x^2yz + h_2(x, z), \quad \checkmark \\f(x, y, z) &= x^2yz + h_3(x, y),\end{aligned}$$

so that $f(x, y, z) = x^2yz$ \checkmark is a potential of the vector field \mathbf{F} .

Absence of minor mistakes. \checkmark