



Université d'Ottawa • University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2122, Fall 2019 – Final exam (practice)

Instructor: A. Tikuisis

Solutions

Read carefully:

- Cellular phones, electronic devices (including calculators) or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.
- This is a closed book exam containing **5 questions**.
- There are two additional blank pages at the end of this exam that you may use as scrap paper. If you run out of space, you may use this page or the backs of pages. Clearly indicate where to find your answer.
- Do not detach the pages of this test, apart from the last (blank) page. If you detach the last page, **do not** use it for your submitted answers.
- You must give clear and complete solutions, with calculations, explanations and justifications. Make sure that your answer is clearly indicated; you must convince me that you understand your solution in order to receive full marks.

By signing below, you acknowledge that you are required to respect the above statements.

Signature: _____

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	Total
Mark						
Out of	21	16	30	15	18	100

1. Multiple choice. Use the following table to record your answers. Write “A”, “B”, “C”, “D”, or “E” to indicate that you have chosen that response, or write “X” to indicate blank (no response). A correct solution is worth **3 marks**, an incorrect or blank solution is worth **0 marks**, and “X” (intentional blank) is worth **1 mark**.

Question part	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
Response							

(i) Let $A := \{(x, y, z) : x \in [-y, z], y \in [0, z], z \in [1, 2]\}$ be an object with density function $\delta(x, y, z) := \frac{2}{y+z}$. What is the mass of A ?

(A) $2 \log(4) - 3 \log(2)$.

(B) 1.

(C) 3.

(D) $\frac{7}{2}$.

(E) 7.

Solution: (C).

$$\begin{aligned} \text{mass} &= \int \int \int_A \delta(x, y, z) \, dx \, dy \, dz = \int_1^2 \int_0^z \int_{-y}^z \frac{2}{y+z} \, dx \, dy \, dz \\ &= \int_1^2 \int_0^z 2 \, dy \, dz = \int_1^2 2z \, dz = 3. \end{aligned}$$

(ii) Let $A := \{(u, v) : u \in [0, 1], v \in [0, u^3]\}$ and define $\Phi : A \rightarrow \mathbb{R}^3$ by

$$\Phi(u, v) := \left(u, v, \frac{u^3}{3} + v\right).$$

What is the surface area of $\Phi(A)$?

(A) $\frac{3\sqrt{2}}{5}$.

(B) $\frac{1}{4}$.

(C) $\frac{3^{3/2} - 2^{3/2}}{6}$.

(D) $\frac{5}{8}$.

(E) $\frac{41}{105}$.

Solution: (C). We have $\Phi_u = (1, 0, u^2)$, $\Phi_v = (0, 1, 1)$, so $\Phi_u \times \Phi_v = (-u^2, -1, 1)$ and $\|\Phi_u \times \Phi_v\| = \sqrt{u^4 + 2}$. Thus the surface area is

$$\begin{aligned} \int \int_A \|\Phi_u \times \Phi_v\| \, du \, dv &= \int_0^1 \int_0^{u^3} \sqrt{u^4 + 2} \, dv \, du \\ &= \int_0^1 u^3 \sqrt{u^4 + 2} \, du = \frac{1}{4} \int_2^3 \sqrt{w} \, dw \\ &= \frac{2}{4 \cdot 3} (3^{3/2} - 2^{3/2}), \end{aligned}$$

using the substitution $w = u^4 + 2$, which has $dw = 4u^3 \, du$.

(iii) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function such that $\nabla f = (y \cos(xy) + \cos(y), x \cos(xy) - x \sin(y))$ and let $g : \mathbb{R} \rightarrow \mathbb{R}^2$ be given by $g(t) := (t^2, t^3)$. What is $(f \circ g)'(t)$?

- (A) $2t^5 \cos(t^5) - t^5 \sin(t^3) + t^2 \cos(t^3)$.
 (B) $12t^3 \cos(6t^3) - 6t^3 \sin(3t^2) + 2t \cos(3t^2)$.
 (C) $(t^3 + t^2) \cos(t^5) - t^2 \sin(t^3) + \cos(t^3)$.
 (D) $(3t^2 + 2t) \cos(6t^3) - 2t \sin(2t^2) + \cos(3t^2)$.
 (E) $5t^4 \cos(t^5) - 3t^4 \sin(t^3) + 2t \cos(t^3)$.

Solution: (E). By the Chain Rule,

$$\begin{aligned} (f \circ g)'(t) &= \nabla f(g(t)) \cdot g'(t) \\ &= (t^3 \cos(t^5) + \cos(t^3), t^2 \cos(t^5) - t^2 \sin(t^3)) \cdot (2t, 3t^2) \\ &= 2t^4 \cos(t^5) + 2t \cos(t^3) + 3t^4 \cos(t^5) - 3t^4 \sin(t^3). \end{aligned}$$

(iv) Let $p : [0, b] \rightarrow \mathbb{R}^2$ be defined by $p(t) := (e^t, t^2)$. What is $\int_p x \, dx + y^2 \, dy$?

- (A) $\frac{e^{2b}-1}{2} + \frac{b^7}{7}$.
 (B) $e^b - 1 + \frac{b^3}{3}$.
 (C) $b^2 e^b - 2b e^b + 2e^b + \frac{1}{2} b^2 e^b - \frac{1}{2} b e^b + \frac{1}{4} e^b - \frac{9}{4}$.
 (D) $\frac{e^{2b}-1}{2} + \frac{b^6}{3}$.
 (E) $\frac{1}{2} b^2 e^{2b} + \frac{1}{3} b^6 e^b$.

Solution: (D). $p'(t) = (e^t, 2t)$. The vector field is $\vec{F} = (x, y^2)$, so $\vec{F}(p(t)) = (e^t, t^4)$. Thus, $\vec{F}(p(t)) \cdot p'(t) = e^{2t} + 2t^5$ and

$$\int_p x dx + y^2 dy = \int_0^b e^{2t} + 2t^5 dt = \frac{e^{2b} - 1}{2} + \frac{b^6}{3}.$$

(v) Define $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $\Phi(u, v) := (u + \frac{v^2}{2}, v, -uv)$ and let S be the surface parametrized by Φ . At which point is the tangent plane to S parallel to the plane given by $z = 2x - 3y - 2$?

(A) $(3, 2, -2)$.

(B) $(5, -2, 14)$.

(C) $(\frac{3}{2}, 1, -1)$.

(D) $(2, -2, 0)$.

(E) $(7, 4, 4)$.

Solution: (B). Rewriting the given plane equation as $(2, -3, -1) \cdot (x, y, z) = 2$, we see that a vector orthogonal to the plane is $(2, -3, -1)$. We compute $\Phi_u \times \Phi_v = (1, 0, -v) \times (v, 1, -u) = (v, u - v^2, 1)$. This gives a vector that is orthogonal to the tangent plane, so for the tangent plane and the given plane to be parallel, these two vectors need to be in the same (or opposite) directions, i.e., we must solve

$$(v, u - v^2, 1) = \lambda(2, -3, -1).$$

The third component implies $\lambda = -1$. Using this and the first component, we get $v = -2$. Then the second component becomes $u - 2^2 = 3$ which solves as $u = 7$. Thus the tangent plane is parallel to this plane at the point

$$\Phi(7, -2) = (5, -2, 14).$$

(vi) Let $A, B \subseteq \mathbb{R}^3$ be objects with uniform density δ , such that

$$B = \{(2x, 2y, 2z) : (x, y, z) \in A\}.$$

Suppose that the moment of inertia of A about the z -axis is K . What is the moment of inertia of B about the z -axis?

- (A) K .
- (B) $\frac{K}{2}$.
- (C) $2K$.
- (D) $32K$.
- (E) $4\sqrt{2}K$.

Solution: (D). We have $K = \int \int \int_A x^2 + y^2 \delta \, dx \, dy \, dz$ and the answer is $\int \int \int_B x^2 + y^2 \delta \, dx \, dy \, dz$. Use change of variables with $\Phi(x, y, z) := (2x, 2y, 2z)$, so that $\Phi(A) = B$. One computes $\det D\Phi = 8$, so

$$\int \int \int_B x^2 + y^2 \, dx \, dy \, dz = \int \int \int_A ((2x)^2 + (2y)^2) 8 \, dx \, dy \, dz = 32K.$$

(vii) Let $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a C^2 vector field. In order to prove that $\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$, which of the following is used?

- (A) Equality of mixed partial derivatives.
- (B) Green's Theorem.
- (C) Fubini's Theorem.
- (D) The Chain Rule.
- (E) All of the above.

Solution: (A).

2. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) := xe^y + y \cos(\pi x) - \frac{x^2}{2}.$$

(i) Determine the Taylor polynomial of degree 2 centred at $(0, 0)$. 10

Solution: $f(0, 0) = 0$.

$\frac{\partial f}{\partial x} = e^y - \pi y \sin(\pi x) - x$ and $\frac{\partial f}{\partial y} = xe^y + \cos(\pi x)$. So at $(0, 0)$ these are 1 and 1.

$\frac{\partial^2 f}{\partial x^2} = -\pi^2 y \cos(\pi x) - 1$, $\frac{\partial^2 f}{\partial x \partial y} = e^y - \pi \sin(\pi x)$, and $\frac{\partial^2 f}{\partial y^2} = xe^y$. So at $(0, 0)$ these are $-1, 1$, and 0 . Thus the second degree Taylor polynomial is

$$P_2(s, t) = s + t - \frac{s^2}{2} + st.$$

(ii) Prove that f has a critical point at $(1, 0)$, and determine whether this critical point is a local maximum, a local minimum, or a saddle point. 6

Solution: Using the calculations from (i), we have

$$\nabla(1, 0) = (1 - 0 - 1, 1 - 1) = (0, 0),$$

and thus $(1, 0)$ is a critical point.

Also using the calculations from (i), the Hessian at $(1, 0)$ is

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The determinant is $-2 < 0$, so by the second derivative test, $(1, 0)$ is a saddle point.

3. Let $A := \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ and define $\vec{F} : A \rightarrow \mathbb{R}^3$ by

$$\vec{F}(x, y, z) := (x^3 + 3yz, y^3 + e^z, z^3 - \frac{y}{1+x^4}).$$

(i) What is the boundary of A , ∂A ?

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Solution: $\partial A = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$.

(ii) Write out a parametrization $\Phi : D \rightarrow \partial A$ for ∂A such that the vector $\Phi_u \times \Phi_v$ points away from the origin at the point $\Phi(u, v)$, for all u, v .

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Solution: Define $\Phi(u, v) := (\cos v \sin u, \sin v \sin u, \cos u)$. Then

$$\begin{aligned}\Phi_u \times \Phi_v &= (\cos v \cos u, \sin v \cos u, -\sin u) \times (-\sin v \sin u, \cos v \sin u, 0) \\ &= \sin u (\cos v \sin u, \sin v \sin u, \cos u) = \sin u \Phi(u, v).\end{aligned}$$

This points away from the origin, and is outward-pointing.

(iii) Write out an expression for $\int \int_{\Phi} \vec{F} \cdot d\vec{S}$. That is, write it as $\int \int_D f \, dx \, dy$ for some function $f : D \rightarrow \mathbb{R}$. You should fully expand f , i.e., the final answer should not include Φ or \vec{F} . Do not solve this integral. 6

Solution:

$$\begin{aligned} \int \int_{\Phi} \vec{F} \cdot d\vec{S} &= \int \int_D \vec{F}(\Phi(u, v)) \cdot (\Phi_u \times \Phi_v) \, du \, dv \\ &= \int \int_D \sin u ((\cos v \sin u)^3 + 3 \sin v \sin u \cos u, (\sin v \sin u)^3 + e^{\cos u}, \\ &\quad (\cos u)^3 - \frac{\sin v \cos u}{1 + (\cos v \sin u)^4}) \cdot (\cos v \sin u, \sin v \sin u, \cos u) \, du \, dv \\ &= \int \int_D (\cos v)^4 (\sin u)^5 + 3 \sin v^2 \sin u \cos u \cos v \sin u \\ &\quad + (\sin v)^4 (\sin u)^5 + \sin u^2 \sin v e^{\cos u}, \sin u (\cos u)^4 \\ &\quad - \frac{\cos u^2 \sin u}{1 + (\cos v \sin u)^4}) \, du \, dv \end{aligned}$$

(iv) Determine $\text{div}(\vec{F})$.

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Solution:

$$\begin{aligned} \text{div}(\vec{F}) &= \frac{\partial}{\partial x}(x^3 + 3yz) + \frac{\partial}{\partial y}(y^3 + e^z) + \frac{\partial}{\partial z}(z^3 - \frac{y}{1+x^4}) \\ &= 3x^2 + 3y^2 + 3z^2. \end{aligned}$$

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(v) Using Gauss' Divergence Theorem, compute $\int \int_{\Phi} \vec{F} \cdot d\vec{S}$.

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Solution: Noting that Φ has the outward orientation, Gauss' Divergence Theorem implies

$$\begin{aligned} \int \int_{\Phi} \vec{F} \cdot d\vec{S} &= \int \int \int_A \operatorname{div}(\vec{F}) \, dx \, dy \, dz \\ &= \int \int \int_A 3(x^2 + y^2 + z^2) \, dx \, dy \, dz \end{aligned}$$

To integrate over A , we do change-of-variables with spherical coordinates, to get

$$\begin{aligned} \int_0^1 \int_0^{2\pi} \int_0^{\pi} 3\rho^2 \cdot \rho^2 \sin(\phi) \, d\phi \, d\theta \, d\rho &= \int_0^1 \int_0^{2\pi} 3\rho^4 (-\cos(0) + \cos(\pi)) \, d\theta \, d\rho \\ &= \int_0^1 12\pi\rho^4 \, d\rho = \frac{12\pi}{5}. \end{aligned}$$

4. Find the minimum and maximum of the function $f(x, y) := e^{x+y}$ on the set $\{(x, y) : e^x + e^{2y} = 3e, x, y \geq 0\}$.

You may use the fact that $2e^{3/2} > 3e - 1 > 1$.

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Solution: We use Lagrange Multipliers. The given set is the level set of $g(x, y) := e^x + e^{2y}$. We compute

$$\nabla f = (e^{x+y}, e^{x+y}) \quad \text{and} \quad \nabla g = (e^x, 2e^{2y}).$$

We solve

$$\begin{aligned} e^{x+y} &= \lambda e^x, \\ e^{x+y} &= \lambda 2e^{2y}, \\ e^x + e^{2y} &= 3e. \end{aligned}$$

We see from the first equations that $\lambda \neq 0$. Then it follows that $e^x = 2e^{2y} = e^{\log(2)+2y}$, so the last equation becomes $3e^{2y} = 3e$. This solves as $y = \frac{1}{2}$, which gives $x = 1 + \log(2)$. We have $f(1 + \log(2), \frac{1}{2}) = e^{3/2+\log(2)} = 2e^{3/2}$.

There are two boundary points of the set: one where $x = 0$ and the other where $y = 0$. If $x = 0$, then $e^{2y} = 3e - 1$,

$$f(x, y) = e^{x+y} = e^y = \sqrt{e^{2y}} = \sqrt{3e - 1}.$$

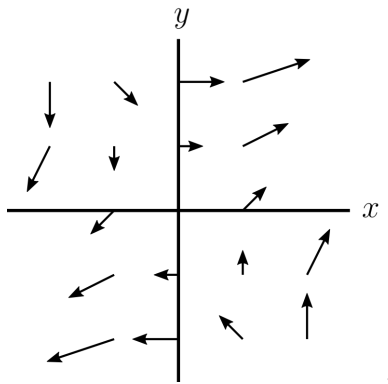
If $y = 0$, then $e^x = 3e - 1$ so $f(x, y) = e^{x+y} = e^x = 3e - 1$.

Since $3e - 1 > 1$, it follows that $\sqrt{3e - 1} < 3e - 1$. Since $2e^{3/2} > 3e - 1$, the maximum is $2e^{3/2}$. The minimum is $\sqrt{3e - 1}$.

5. Consider the vector field $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $\vec{F}(x, y) = (x + y, x)$.

(i) Sketch this vector field.

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(ii) Compute the scalar curl of \vec{F} .

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Solution: The scalar curl is $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - 1 = 0$.

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(iii) Does there exist a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\nabla f = \vec{F}$? If so, write down such a function f . 8

Solution:

By a theorem, since the scalar curl is 0, it follows that there is such a function.

Such a function can be given by $f(x, y) := \int_p \vec{F} \cdot d\vec{s}$, where p is a path from $(0, 0)$ to (x, y) . Use the path $p(t) := (tx, ty)$ for $t \in [0, 1]$. Then $p'(t) = (x, y)$ and $\vec{F}(p(t)) = (tx + ty, tx)$, so

$$f(x, y) = \int_p \vec{F} \cdot d\vec{s} = \int_0^1 (tx + ty, tx) \cdot (x, y) dt = \int_0^1 tx^2 + txy + txy dt = \frac{x^2}{2} + xy.$$

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