Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2122, Fall 2019 – Midterm exam 3 (practice) **Instructor: A. Tikuisis**

Solutions

Read the following instructions:

- The use of cellphones, electronic devices (including calculators), and course notes is strictly forbidden. All phones and electronic devices must be turned off and kept in your bags: do not leave them on you. If you are seen to have an electronic device on your person, we may ask you to leave the exam immediately, and fraud allegations could be made, which could lead to a mark of 0 (zero) on this midterm.
- The duration of this midterm is 75 minutes.
- This is a closed book midterm containing **5 questions**.
- There is an additional blank page at the end of this exam that you may use as scrap paper. If you run out of space, you may use this page or the backs of pages. Clearly indicate where to find your answer.
- Do not detach the pages of this test, apart from the last (blank) page. If you detach the last page, do not use it for your submitted answers.
- You must give clear and complete solutions, with calculations, explanations and justifications. Make sure that your answer is clearly indicated; you must convince me that you understand your solution in order to receive full marks.

By signing below, you acknowledge that you are required to respect the above statements.

Signature: ____________________________

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	Total
Mark						
Out of	9	12	10	8	10	50

1. Multiple choice. Write your answer clearly in the blank below the question, or write "X" to indicate blank. Each question is worth **3 marks** and has exactly one correct answer. A correct solution is worth 3 marks, an incorrect or blank solution is worth 0 marks, and "X" (intentional blank) is worth 1 mark.

(i) Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field and suppose that

$$D\vec{F}(0,0,0) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}.$$

What is $\text{div}(\vec{F})(0,0,0)$?

(A) 0.

(B) 8

(C) 21.

(D) $(3,6,12)$.

(E) $(6,7,8)$.

Solution: (B).

Writing $\vec{F} = (F_1, F_2, F_3)$, we have $\text{div}(\vec{F}) = \frac{F_1}{\partial x} + \frac{F_2}{\partial y} + \frac{F_3}{\partial z} = 1 + 2 + 5 = 8$.

(ii) Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^1 function and $c : [0,1] \to \mathbb{R}^3$ is a C^1 path, such that

$$\begin{aligned} c(0) &= (0,0,0), & c(1) &= (1,1,1) \\ c'(0) &= (1,0,0), & c'(1) &= (2,0,0), \\ f(0,0,0) &= 0, & f(1,0,0) &= 3, \\ f(1,1,1) &= 5, & f(2,0,0) &= 200, \\ \nabla f(0,0,0) &= (10,0,0), & \nabla f(1,1,1) &= (20,0,0). \end{aligned}$$

What is $\int_c \nabla f \cdot d\vec{s}$?

(A) 5.

(B) 30.

(C) 197.

(D) $(10,0,0)$.

(E) $(397,0,0)$.

Solution: (A)

By a theorem, we have

$$\begin{aligned}\int_c \nabla f \cdot d\vec{s} &= f(c(1)) - f(c(0)) \\ &= f(1,1,1) - f(0,0,0) = 5 - 0.\end{aligned}$$

(iii) Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 vector field and suppose that

$$D\vec{F}(0,0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

What is $\text{div}(\text{curl}(\vec{F}))(0,0,0)$?

(A) -1.

(B) 0.

(C) 12.

(D) $(1,4,7)$.

(E) $(4,3,5)$.

Solution: (B).

By a theorem, $\text{div}(\text{curl}(\vec{F}))$ is always 0 provided $\vec{F}$ is a C^2 vector field.

2. Let S be a surface parametrized by $\Phi(u,v) := (u\cos(v), u\sin(v), v)$ where $u \in [0,1]$ and $v \in [0,2\pi]$.

(i) Determine $\Phi_u \times \Phi_v$ and show that $\|\Phi_u \times \Phi_v\| = \sqrt{1+u^2}$. 6

Solution: We have

$$\Phi_u = (\cos(v), \sin(v), 0), \quad \Phi_v = (-u\sin(v), u\cos(v), 1).$$

Thus,

$$\begin{aligned}\Phi_u \times \Phi_v &= (\sin(v), -\cos(v), \cos(v)(u\cos(v)) - \sin(v)(-u\sin(v))) \\ &= (\sin(v), -\cos(v), u).\end{aligned}$$

Consequently,

$$\begin{aligned}\|\Phi_u \times \Phi_v\| &= \sqrt{(\sin(v))^2 + (-\cos(v))^2 + u^2} \\ &= \sqrt{1+u^2}.\end{aligned}$$

(ii) Suppose that this surface has a density function given by $\delta(x,y,z) := 2z\sqrt{x^2+y^2}$ (in grams/unit^2). Determine the total mass (in grams) of the surface.

(Hint. Show that $\delta(\Phi(u,v)) = 2uv$.) 6

Solution: We have

$$\delta(\Phi(u,v)) = \delta(u\cos(v), u\sin(v), v) = 2v\sqrt{(u\cos(v))^2 + (u\sin(v))^2} = 2v\sqrt{u^2} = 2uv.$$

Thus, the mass is given by

$$\begin{aligned}\int\int_S \delta\, dS &= \int_0^1 \int_0^{2\pi} 2uv\sqrt{1+u^2}\, dv\, du \\ &= \int_0^1 \frac{(2\pi)^2}{2} 2u\sqrt{1+u^2}\, du.\end{aligned}$$

We solve this by substituting $w = 1+u^2$, so $dw = 2u$, so

$$\int 2u\sqrt{1+u^2}, du = \int \sqrt{w}\, dw = \frac{2}{3}w^{3/2} = \frac{2}{3}(1+u^2)^{3/2}.$$

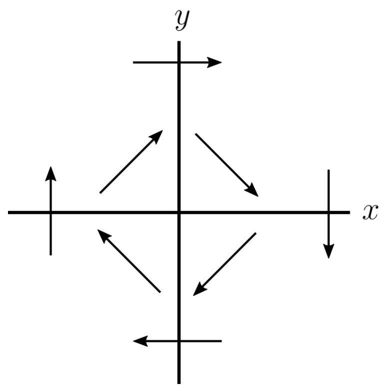
Thus, the mass is

$$\begin{aligned}\int_0^1 \frac{(2\pi)^2}{2} 2u\sqrt{1+u^2}\, du &= 2\pi^2 \frac{2}{3}\Big((1+1^2)^{3/2} - (1+0^2)^{3/2}\Big) \\ &= \frac{4\pi^2(2\sqrt{2}-1)}{3}.\end{aligned}$$

3. Define the vector field $\vec{F}(x,y) := (\frac{y}{\sqrt{x^2+y^2}}, \frac{-x}{\sqrt{x^2+y^2}})$.

(i) Sketch this vector field in the space provided. 5

Solution:



We note that $\|\vec{F}(x,y)\| = \sqrt{(\frac{y}{\sqrt{x^2+y^2}})^2 + (\frac{-x}{\sqrt{x^2+y^2}})^2} = 1$, so at every point $\vec{F}(x,y)$ is a unit vector.

(ii) Define a path $p : [0,\pi] \to \mathbb{R}^2$ by

$$p(t) := (\cos(t), \sin(t)).$$

Compute $\int_p \vec{F} \cdot d\vec{s}$. 5

Solution: We compute $p'(t) = (-\sin(t), \cos(t))$ and $\vec{F}(p(t)) = (\sin(t), -\cos(t))$, so

$$\vec{F}(p(t)) \cdot p'(t) = \sin(t)(-\sin(t)) - \cos(t)\cos(t) = -1.$$

Thus,

$$\int_p \vec{F} \cdot d\vec{s} = \int_0^\pi -1\, dt = -\pi.$$

4. Let A be the body $[-1,1] \times [0,3] \times [-2,2]$, and assume that its density function is given by $\delta(x,y,z) = 1$ (i.e., its mass is uniformly distributed).

(i) Determine the moments of inertia about the y-axis and the z-axis. 7

Solution:

$$\begin{aligned}
I_y &= \int_{-1}^{1}\int_{0}^{3}\int_{-2}^{2} x^2 + z^2 \, dz\, dy\, dx \\
&= \int_{-1}^{1}\int_{-2}^{2} 3x^2 + 3z^2 \, dz\, dx \\
&= \int_{-1}^{1} 12x^2 + 16 \, dx \\
&= 8 + 32 = 40
\end{aligned}$$

and

$$\begin{aligned}
I_z &= \int_{-1}^{1}\int_{0}^{3}\int_{-2}^{2} x^2 + y^2 \, dz\, dy\, dx \\
&= 4\int_{-1}^{1}\int_{0}^{3} x^2 + y^2 \, dy\, dx \\
&= 4\int_{-1}^{1} 3x^2 + 9 \, dx \\
&= 4(2 + 18) = 80.
\end{aligned}$$

(ii) Would it be harder to cause A to rotate about the y-axis or to rotate (the same amount) about the z-axis? 1

Solution: The z-axis, since its moment of inertia is greater.

5. Define a path $p(t) := (t, e^{2t}, 2e^t)$ for $t \in [0, 2]$.

(i) Compute $p'(t)$ and show that $\|p'(t)\| = 1 + 2e^{2t}$. 4

Solution: $p'(t) = (1, 2e^{2t}, 2e^t)$, so

$$\begin{aligned}\|p'(t)\| &= \sqrt{1 + (2e^{2t})^2 + (2e^t)^2} \\ &= \sqrt{1 + 4e^{4t} + 4e^{2t}} \\ &= \sqrt{(1 + 2e^{2t})^2} = 1 + 2e^{2t}.\end{aligned}$$

(ii) Compute the path length of p. 6

Solution: The path length is

$$\begin{aligned}\int_p 1\,ds &= \int_0^2 1 + 2e^{2t}\,dt \\ &= t + e^{2t}\Big|_{t=0}^2 \\ &= (2 + e^4) - (0 + e^0) \\ &= 1 + e^4.\end{aligned}$$

(The last step is not necessary.)

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