Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2122, Fall 2019 – Midterm exam 1 (practice) **Instructor: A. Tikuisis**

Solutions

Read the following instructions:

- The use of cellphones, electronic devices (including calculators), and course notes is strictly forbidden. All phones and electronic devices must be turned off and kept in your bags: do not leave them on you. If you are seen to have an electronic device on your person, we may ask you to leave the exam immediately, and fraud allegations could be made, which could lead to a mark of 0 (zero) on this midterm.
- The duration of this midterm is 75 minutes.
- This is a closed book midterm containing **5 questions**.
- Do not detach the pages of this test.
- There is an additional blank page at the end of this exam that you may use as scrap paper. If you run out of space, you may use this page or the back of pages. Clearly indicate where to find your answer if it is not entirely contained in the space provided on the page.
- You must give clear and complete solutions, with calculations, explanations and justifications. Make sure that your answer is clearly indicated; you must convince me that you understand your solution in order to receive full marks.

By signing below, you acknowledge that you are required to respect the above statements.

Signature: ____________________________

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	Total
Mark						
Out of	9	10	11	10	10	50

1. Multiple choice. Please circle your answer, or in case you change your mind, write your answer clearly below the question. Each question is worth **3 marks**. A correct solution is worth 3 marks, an incorrect solution is worth 0 marks, and no solution is worth 1 mark.

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be function which is differentiable at $(0, 0)$. Which of the following is correct:

(i) $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ both exist, but f might not be continuous at $(0, 0)$.

(ii) $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ both exist and f is continuous at $(0, 0)$.

(iii) $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and these partial derivative functions are continuous at $(0, 0)$.

(iv) $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ both exist and are equal.

(v) None of the above.

Solution. (ii).

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function, and suppose that

$$\nabla f(1,2) = (0,0), \quad Hf(1,2) = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix},$$

$$\nabla f(3,4) = (5,-5), \quad Hf(3,4) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Which of the following is correct:

(i) $(1, 2)$ is a local minimum of f and $(3, 4)$ is neither a local minimum nor a local maximum.

(ii) Both $(1, 2)$ and $(3, 4)$ are local maxima of f.

(iii) $(1, 2)$ is a local maximum of f and $(3, 4)$ is neither a local minimum nor a local maximum.

(iv) $(1, 2)$ is neither a local minimum nor a local maximum, and the same is true of $(3, 4)$.

(v) None of the above.

Solution. (iv).

$\det(Hf(1, 2)) = 3 - 4 = -1 < 0$, so $(1, 2)$ is neither a local min nor max by the second derivative test.

$(3, 4)$ is neither a local min nor max by the first derivative test.

(c) Let $\vec{v}, \vec{w} \in \mathbb{R}^n$. Then $(\vec{v} + \vec{w}) \times (\vec{v} - \vec{w})$ is equal to:

(i) $2\vec{v} \times \vec{w}$.

(ii) $\vec{v} \times \vec{v} + 2\vec{v} \times \vec{w} + \vec{w} \times \vec{w}$.

(iii) $\vec{v} \times \vec{v} + \vec{v} \times \vec{w}$.

(iv) $\vec{0}$.

(v) None of the above.

Solution. (v).

$$\begin{aligned}
(\vec{v} + \vec{w}) \times (\vec{v} - \vec{w}) &= \vec{v} \times \vec{v} - \vec{v} \times \vec{w} + \vec{w} \times \vec{v} - \vec{w} \times \vec{w} \\
&= \vec{0} - \vec{v} \times \vec{w} - \vec{v} \times \vec{w} - \vec{0} \\
&= -2\vec{v} \times \vec{w}
\end{aligned}$$

2. Define $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\phi(x, y) = (x, x - y)$ and define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(u, v) := u\cos(v) + u^2 v$.

(a) Compute the gradient $\nabla f(u, v)$. (2)

(b) Compute $D\phi(x, y)$. (2)

(c) Using the Chain Rule, show that $\nabla(f \circ \phi)(2, 2) = (5, -4)$. (3)

(d) Using (c), give a formula for the tangent plane of $f \circ \phi$ at $(2, 2)$. (3)

Solution. (a): $\nabla f(u, v) = (\cos(v) + 2uv, -u\sin(v) + u^2)$.

(b): $D\phi(x, y) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$.

(c): First compute $\phi(2, 2) = (2, 0)$. As row vectors, we have

$$\begin{aligned} \nabla(f \circ \phi)(2, 2) &= \nabla f(\phi(2, 2))D\phi(2, 2) \\ &= \nabla f(2, 0)D\phi(2, 2) \\ &= \begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -4 \end{bmatrix}. \end{aligned}$$

(d): First compute $f(\phi(2, 2)) = f(2, 0) = 2$. The tangent plane is given by

$$\begin{aligned} z &= f(\phi(2, 2)) + \nabla(f \circ \phi)(2, 2) \cdot (x - 2, y - 2) \\ &= 2 + (5, -4) \cdot (x - 2, y - 2) \\ &= 2 + 5(x - 2) - 4(y - 2) \\ &= 5x - 4y \end{aligned}$$

(*Note. It is not necessary to perform the last step:* $z = 2 + 5(x - 2) - 4(y - 2)$ *is considered a correct answer.*)

3. Define $c : \mathbb{R} \to \mathbb{R}^2$ by $c(t) := (t^2, 1 + t^3)$ and $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x^3 - y^2 + 2y.$$

Let S be the level set of f at $k = 1$.

(a) Write the definition of the level set of f at k. (2)

(b) Prove that the curve defined by $c(t)$ is contained in S. (3)

(c) Give a formula for the tangent line to c at $t = 1$. (3)

(d) Give a formula for the tangent line to S at the point $(1, 2)$. (3)

Solution. (a): $S := \{(x, y) \in \mathbb{R}^2 : f(x, y) = k\}$.

(b): We need to show that $f(c(t)) = 1$ for all $t \in \mathbb{R}$. We have

$$f(c(t)) = f(t^2, 1+t^3) = (t^2)^3 - (1+t^3)^2 + 2(1+t^3) = t^6 - 1 - 2t^3 - t^6 + 2 + 2t^3 = 1.$$

(c): Compute

$$c(1) = (1, 2), \quad c'(t) = (2t, 3t^2) \quad c'(1) = (2, 3).$$

Hence the tangent line is given by

$$l(t) = c(1) + tc'(1) = (1, 2) + t(2, 3) = (1 + 2t, 2 + 3t).$$

(*Note. It is not necessary to perform the last step:* $l(t) = (1, 2) + t(2, 3)$ *is considered a correct answer.*)

(d): Compute $\nabla f(x, y) = (3x^2, -2y + 2)$. The tangent line to S at this point is orthogonal to $\nabla f(1, 2) = (3, -2)$. So its equation is

$$0 = \nabla f(1, 2) \cdot ((x, y) - (1, 2)) = (3, -2) \cdot (x - 1, y - 2) = 3(x - 1) - 2(y - 2) = 3x - 2y + 1.$$

(*Note. It is not necessary to perform the last step:* $0 = 3(x - 1) - 2(y - 2)$ *is considered a correct answer.*)

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) := e^{xy}$.

(a) Find the second order Taylor polynomial of f centred at $(0, 2)$. (5)

(b) Determine the intersection of the tangent plane to f at $(0, 2)$ with the x-axis. (5)

Solution. (a): Compute

$$\begin{aligned} f(0,2) &= e^0 = 1. \\ \frac{\partial f}{\partial x} &= ye^{xy}, & \frac{\partial f}{\partial x}(0,2) &= 2e^0 = 2 \\ \frac{\partial f}{\partial y} &= xe^{xy}, & \frac{\partial f}{\partial y}(0,2) &= 0e^0 = 0 \\ \frac{\partial^2 f}{\partial x^2} &= y^2 e^{xy}, & \frac{\partial^2 f}{\partial x^2}(0,2) &= 4e^0 = 4 \\ \frac{\partial^2 f}{\partial x \partial y} &= e^{xy} + xye^{xy}, & \frac{\partial^2 f}{\partial x \partial y}(0,2) &= e^0 + 0e^0 = 1 \\ \frac{\partial^2 f}{\partial y^2} &= x^2 e^{xy}, & \frac{\partial^2 f}{\partial y^2}(0,2) &= 0e^0 = 0. \end{aligned}$$

Thus the 2nd order Taylor polynomial is

$$\begin{aligned} P_2(0+h, 2+k) &= f(0,2) + \frac{\partial f}{\partial x}(0,2)h + \frac{\partial f}{\partial y}(0,2)k \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,2)h^2 + \frac{\partial^2 f}{\partial x \partial y}(0,2)hk + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,2)k^2 \\ &= 1 + 2h + 0k + \tfrac{1}{2}4h^2 + 1hk + 0k^2 \\ &= 1 + 2h + 2h^2 + hk \end{aligned}$$

(b): The tangent plane at $(0, 2)$ is

$$P_1(0+h, 2+k) = f(0,2) + \frac{\partial f}{\partial x}(0,2)h + \frac{\partial f}{\partial y}(0,2)k = 1 + 2h.$$

As an equation in (x, y, z) this is

$$z = P_1(x, 2 + (y-2)) = 1 + 2x.$$

The x-axis is $\{(x, 0, 0) : x \in \mathbb{R}\}$, so we set $y = z = 0$ and solve for x:

$$0 = 1 + 2x, \quad x = -\frac{1}{2}.$$

Thus the intersection with the x-axis is $(-\frac{1}{2}, 0, 0)$.

5. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions such that $f(x,y) - g(y,x)$ is constant.

(a) Prove that

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial g}{\partial y}(a,b), \quad \text{for all } (a,b) \in \mathbb{R}^2.$$

(5)

(b) Prove *using the definition of local minimum* that if f has a local minimum at $(3,3)$ then so does g. (5)

Solution. (a): Suppose that $f(x,y) - g(y,x) = k$ for all x, y, i.e., $f(x,y) = k + g(y,x)$. We have

$$\begin{aligned}\frac{\partial f}{\partial x}(a,b) &= \lim_{x\to a} \frac{f(x,b) - f(a,b)}{x-a} \\ &= \lim_{x\to a} \frac{k + g(b,x) - (k + g(b,a))}{x-a} \\ &= \lim_{x\to a} \frac{g(b,x) - g(b,a)}{x-a} \\ &= \lim_{y\to a} \frac{g(b,y) - g(b,a)}{y-a} \\ &= \frac{\partial g}{\partial y}(b,a).\end{aligned}$$

(b): The definition says that there is some $r > 0$ such that for all $(x,y) \in \mathbb{R}^2$, if $\|(x,y) - (3,3)\| < r$ then $f(3,3) \leq f(x,y)$. We want to show the same holds for g. So, for $(x,y) \in \mathbb{R}^2$, if $\|(x,y) - (3,3)\| < r$ then we can rearrange this as $\sqrt{(x-3)^2 + (y-3)^2} < r$ or equivalently $\|(y,x) - (3,3)\| < r$, so

$$g(3,3) = f(3,3) - k \leq f(y,x) - k = g(x,y).$$

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