Tutorial #6 - Questions

Question 1:

Use logarithmic differentiation to derive the following function. Make sure you isolate for y' and write y' as a function of x.

$$y = \frac{(e^{x^2+3x})(x^5 - 2x^4 + 1)^{10}}{(2x+1)^{10}(1-x)^{12}}$$

Solution:

Taking the ln on both sides we get:

$$\ln(y) = \ln\left(\frac{(e^{x^2+3x})(x^5 - 2x^4 + 1)^{10}}{(2x+1)^{10}(1-x)^{12}}\right)$$

$$\ln(y) = \ln\left((e^{x^2+3x})\right) + \ln\left((x^5 - 2x^4 + 1)^{10}\right) - \ln\left((2x+1)^{10}\right) - \ln\left((1-x)^{12}\right)$$

$$\ln(y) = x^2 + 3x + 10\ln(x^5 - 2x^4 + 1) - 10\ln(2x+1) - 12\ln(1-x)$$

Now we can derive implicitly to get:

$$\frac{1}{y}y' = 2x + 3 + \frac{10}{x^5 - 2x^4 + 1}(5x^4 - 8x^3) - \frac{10}{(2x+1)}(2) - \frac{12}{1-x}(-1)$$

$$\frac{1}{y}y' = 2x + 3 + \frac{10(5x^4 - 8x^3)}{x^5 - 2x^4 + 1} - \frac{20}{(2x+1)} + \frac{12}{1-x}$$

$$y' = \left(2x + 3 + \frac{10(5x^4 - 8x^3)}{x^5 - 2x^4 + 1} - \frac{20}{(2x+1)} + \frac{12}{1-x}\right)y$$

$$y' = \left(2x + 3 + \frac{10(5x^4 - 8x^3)}{x^5 - 2x^4 + 1} - \frac{20}{(2x+1)} + \frac{12}{1-x}\right)\left(\frac{(e^{x^2+3x})(x^5 - 2x^4 + 1)^{10}}{(2x+1)^{10}(1-x)^{12}}\right)$$

Question 2:

Use logarithmic differentiation to derive the following function. Make sure you isolate for y' and write y' as a function of x : $y = (x^2 + \ln(3x))^{\cos(x)}$

Solution:

Taking the ln on both sides we get:

$$\ln(y) = \ln\left((x^2 + \ln(3x))^{cos(x)}\right)$$

$$\ln(y) = Cos(x)\ln(x^2 + \ln(3x))$$

Now we can derive implicitly to get:

$$\frac{1}{y}y' = -Sin(x)\ln(x^2 + \ln(3x)) + Cos(x)\frac{1}{x^2 + \ln(3x)}\left(2x + \frac{3}{3x}\right)$$

$$\frac{1}{y}y' = -Sin(x)\ln(x^2 + \ln(3x)) + \frac{Cos(x)}{x^2 + \ln(3x)}\left(\frac{2x^2 + 1}{x}\right)$$

$$y' = \left(-Sin(x)\ln(x^2 + \ln(3x)) + \frac{(2x^2 + 1)Cos(x)}{(x^2 + \ln(3x))(x)}\right)y$$

$$y' = \left(-Sin(x)\ln(x^2 + \ln(3x)) + \frac{(2x^2 + 1)Cos(x)}{x(x^2 + \ln(3x))}\right)(x^2 + \ln(3x))^{\cos(x)}$$

Question 3:

Explain 2 instances when we should use logarithmic differentiation:

Solution:

When the problem calls for a lot of products, quotients, and power rules.

When the problem has $f(x)^{g(x)}$

Question 4:

Determine the linearization of $f(x) = Sin(x)$ at $x = \frac{\pi}{6}$ (use the formula and **do not** expand into y=mx+b)

Solution:

$f(x) = \sin(x) \qquad f'(x) = \cos(x)$

$a = \frac{\pi}{6}$

$f(a) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad f'(a) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$L(x) = f(a) + f'(a)(x - a)$

$L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$

Question 5:

Determine the linearization of $x^{1/3}$, a-value, and c-value to approximate $28^{\frac{1}{3}}$. Then use this to approximate $28^{\frac{1}{3}}$.

Solution:

$$f(x) = x^{1/3}$$
$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$
$$a = 27$$
$$f(a) = 27^{1/3} = 3$$
$$f'(a) = \frac{1}{3}(27)^{-2/3} = \frac{1}{27}$$
$$c = 28$$

This means that:

$$L(x) = f(a) + f'(a)(x - a)$$
$$L(28) = 3 + \frac{1}{27}(28 - 27) = \frac{82}{27}$$
$$f(28) = 28^{1/3} \approx \frac{82}{27} = L(28)$$

Question 6:
Determine the absolute optimum values of the function: $f(x) = x^3 - 12x + 3$ on $[-3,4]$

Solution:
First we derive the function:
$f'(x) = 3x^2 - 12$

We find critical points. Notice that there are no division by 0 in the derivative, this means we only need to find when the derivative is 0.

$0 = 3x^2 - 12$
$12 = 3x^2$
$4 = x^2$
$\pm 2 = x$

Lastly we test all critical points and endpoints to see which is the largest and which is the smallest:
$f(-3) = -27 + 36 + 3 = 12$
$f(-2) = -8 + 24 + 3 = 19$
$f(2) = 8 - 24 + 3 = -13$
$f(4) = 64 - 48 + 3 = 19$

This means the absolute max is 19, and the absolute min is -13.

Question 7:
Determine the intervals of increasing and decreasing: $f(x) = x\ln(x) - 3x$

Solution:
First we derive to find the critical points:

$f'(x) = \ln(x) + \frac{1}{x}x - 3 = \ln(x) - 2$

$0 = \ln(x) - 2$
$2 = \ln(x)$
$x = e^2$

We can see that we don't have any undefined slope that is defined in the domain of the original function (note that the domain of the original function is $D = \{x \in R | x > 0\}$)

We construct an interval table to get:

0 $\qquad$ e^2 $\qquad$ ∞

	1	e^3
$\ln(x) - 2$	$-$	$+$
Total Sign	$-$	$+$

This means that our function is decreasing when $0 < x < e^2$ and is increasing when $x > e^2$

Question 8:

Find all x-values of the critical points and classify them as local max, local min, or neither:

$$f(x) = x^{\frac{1}{3}}e^{3x^2+3x}$$

Solution:

First we derive to find the critical points:

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}e^{3x^2+3x} + x^{\frac{1}{3}}e^{3x^2+3x}(6x+3)$$

$$f'(x) = x^{-\frac{2}{3}}e^{3x^2+3x}\left(\frac{1}{3} + x(6x+3)\right)$$

$$f'(x) = x^{-\frac{2}{3}}e^{3x^2+3x}\left(\frac{1}{3} + 6x^2 + 3x\right)$$

$$f'(x) = \frac{x^{-\frac{2}{3}}e^{3x^2+3x}}{3}(18x^2+9x+1)$$

Here we see that the derivative is undefined when x = 0 due to $x^{-\frac{2}{3}}$, e^{2x^2+5x} is never 0 or undefined, and

$$(18x^2+9x+1) = (6x+1)(3x+1)$$

Which gives $x = -\frac{1}{3}$ or $x = -\frac{1}{6}$

We construct an interval table to get:

	$-\infty$ to $-\frac{1}{3}$	$-\frac{1}{3}$ to $-\frac{1}{6}$	$-\frac{1}{6}$ to 0	0 to ∞
	-1	$-\frac{1}{4}$	$-\frac{1}{8}$	1
$x^{-\frac{2}{3}}$	$+$	$+$	$+$	$+$
e^{3x^2+3x}	$+$	$+$	$+$	$+$
$(6x+1)$	$-$	$-$	$+$	$+$
$(3x+1)$	$-$	$+$	$+$	$+$
Total Sign	$+$	$-$	$+$	$+$

This means that our critical points are when $x = -\frac{1}{3}, -\frac{1}{6}$ and 0.

$x = -\frac{1}{3}$ is a local max

$x = -\frac{1}{6}$ is a local min

$x = 0$ is not a local extremum

Question 9:
Give an example of any function and interval that does not meet the criteria of the Extreme Value Theorem, yet has both an absolute max and an absolute minimum.

Solution:

Many answers exist here:

One could simply change the interval to an open interval, but the extreme points are not the endpoints:

Ex: $y = Sin(x)$ on $(0,2\pi)$

Or you could come up with a discontinuous function that has no vertical asymptotes in the interval and absolute extreme points in the interval:

Ex: $y = x + \frac{x}{x}$ on $[-2,2]$ Will be the line $y = x + 1$ with a hole at x=0, but will still reach its absolute max and min at the endpoints.