MA121 Mock Final Exam

Name: ____

**** Please remember that mock tests are meant as a means of providing an extra set of practice questions and basis for a review class. Do not study for the exam based solely on the topics covered by the mock test! Go back through notes/assignments/homework to ensure you have reviewed all concepts discussed in the course.

Time Allowed: 150 minutes
Total Value: 95 marks
Number of Pages: 8

Instructions:

Cheat Sheet: One 8.5” × 11” page of study notes (both sides) is allowed as a reference while completing the mock test. Please note, that the cheat sheet is permitted for the mock test only!!

No other aids allowed [no calculators].

*Check that your test paper has no missing, blank, or illegible pages. Note that test questions appear on **both** sides of the paper.*

Answer in the spaces provided.

Show all your work. Insufficient justification will result in a loss of marks.

1. [7 *marks*] Let p, q and r be statements.

   (a) Use laws of propositional logic to show that $q \wedge (p \rightarrow (\sim r \rightarrow p))$ is logically equivalent to q.

   (b) Find the disjunctive normal form for the contrapositive of the converse of: $(p \wedge q) \rightarrow r$.

2. [20 *marks*] Prove or disprove each of the following statements. State the method of proof used.

(a) Let $x \in \mathbb{Z}$. If $3 \nmid (x^2 - 1)$, then $3 \mid x$.

(b) If $z_1, z_2 \in \mathbb{C}$, then $|z_1 - z_2| \geq |z_1| - |z_2|$. [Hint: You can use the Triangle Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$]

(c) Let $z \in \mathbb{C}$. If $(\overline{z})^2 = z^2$, then z is either pure real or pure imaginery.

(d) Let $x \in \mathbb{N}$. If $x^2 - x + 2 < 0$, then I will pass my MA121 exam.

#2 Continued: Prove or disprove each of the following statements. State the method of proof used.

(e) Let $x, y \in \mathbb{Z}$. Then $\forall y \; \exists x \; \left[y > 8 \rightarrow x^2 = y \right]$.

(f) Let $m, n \in \mathbb{Z}$. Then $\exists n \; \forall m \; [mn \geq m - n \text{ or } mn \geq n - m]$.

3. [6 *marks*] Use induction on $n \in \mathbb{N}$ to prove that $8 | \left(5^{2n} + 7\right)$ for all $n \geq 1$.

4. [5 *marks*] Use the Binomial Theorem to prove: $\sum_{k=1}^{n} \binom{n}{k} 5^{n-k} (-1)^k = 4^n - 5^n$.

5. [8 *marks*] Two integers a and b are said to be *relatively prime* if $\gcd(a, b) = 1$.

You can assume each of the following statements is true:

1. Two integers a and b are relatively prime, if and only if there are integers k and l such that $ka + lb = 1$.
2. If $m, n, p \in \mathbb{Z}$ such that $p|mn$ with p and m relatively prime, then $p|n$.
3. If $m, n, p \in \mathbb{Z}$ where p is prime and $p|mn$, then either $p|m$ or $p|n$.

Using any of these results where necessary, prove each of the following.

(a) Every two consecutive integers are relatively prime.

(b) If $n, k \in \mathbb{N}$ with $k \leq n$ and $\gcd(k, n) = 1$, then $n \,\Big|\, \binom{n}{k}$.

[Note: You can assume $\binom{n}{k} \in \mathbb{N}$ for all $n, k \in \mathbb{Z}$ such that $n \geq k \geq 0$.]

6. [10 *marks*]

(a) Prove any <u>one</u> of the following statements:

1. If $a, b \in \mathbb{Z}$ and $a \equiv b \pmod{n}$, then $ac \equiv bc \pmod{n}$ for any $c \in \mathbb{Z}$.
2. If $a, b \in \mathbb{Z}$ with $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.
3. If $a, b \in \mathbb{Z}$ and $a \equiv b \pmod{n}$, then $a^2 \equiv b^2 \pmod{n}$.

(b) Find the set of all inverses of 7 modulo 13.

(c) Solve the congruence: $7x \equiv 41 \pmod{13}$.

7. [3 *marks*] Prove that $[x]_{14} \subseteq [x]_7 \ \ \forall x \in \mathbb{Z}$.

8. [5 *marks*] Use modulo arithmetic to determine whether the number $(2\ 345\ 678)^{876\ 543\ 2}+1$ is divisible by 9.

9. [6 *marks*] Use the Euclidean Algorithm and related procedures to determine an integer solution to the equation:

$$576x + 210y = 6.$$

10. [12 *marks*]

(a) Divide the complex numbers $\dfrac{i+4}{5-\sqrt{3}i}$, expressing your final answer in standard form.

(b) Evaluate the following, expressing your final answer in standard form: $\overline{|24-7i| - 2i\,(1+3i)}$

(c) Prove that $\sin(3\theta) = \sin\theta\left(3 - 4\sin^2\theta\right)$ by applying (i) the Binomial Theorem, and (ii) DeMoivre's Theorem, to $(\cos\theta + i\sin\theta)^3$.

11. [6 *marks*] Use either polar form or exponential form, along with DeMoivre's Theorem, to evaluate:

$$\frac{(1+i)^3}{\left(1-\sqrt{3}i\right)^9}.$$

12. [8 *marks*] Find the solution set to the equation $x^4 + 16 = 0$, given that:

(i) $x \in \mathbb{R}$, the set of real numbers;

(ii) $x \in \mathbb{C}$, the set of complex numbers [express your answers in standard form].