~~Lecture 19~~ Lecture 20

Triple integrals

Triple integral is integration over solids (volume) in R^4. For better understand what it is we can see the following table which compares double and triple integrals

Double integrals	Triple integrals
$I=\iint_R f\,dA$	$I=\iiint_e f\,dV$
$R\subset R^2$ is closed region	$e\subset R^3$ is a closed solid
$z=f(x,y)$ is a surface in R^3	$w=f(x,y,z)$ is a 3dim surface in R^3
$I=\iint_R dA$	$I=\iiint_e dV$
The Area of R	The Volume of the solid e

Example 1. Evaluate the triple integral $\iiint_e dv$ where e is defined by

$$e=\{(x,y,z): 0\le z\le xy,\ x^2\le y\le x,\ 0\le x\le 1\}$$

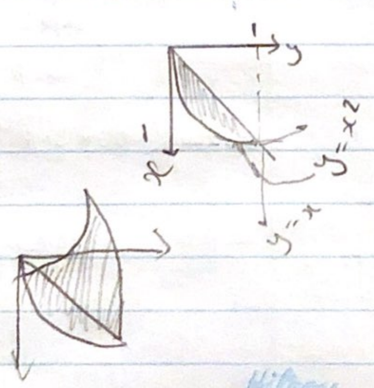
$R=\{(x,y,z)$ so the $x^2\le y\le x,\ 0\le x\le 1$

$0\le z\le xy$

$z=xy$

$z=x^2 y$

$y=x^2 \Rightarrow z=x^3$



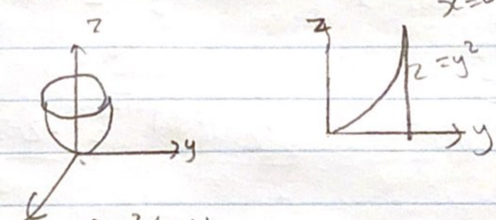
$$\int_0^1\int_{x^2}^{x}\int_0^{xy} dz\,dy\,dx$$

$$=\int_0^1\int_{x^2}^{x}\left(z\Big|_0^{xy}\right)dy\,dx=\int_0^1\int_{x^2}^{x} xy\,dy\,dx$$

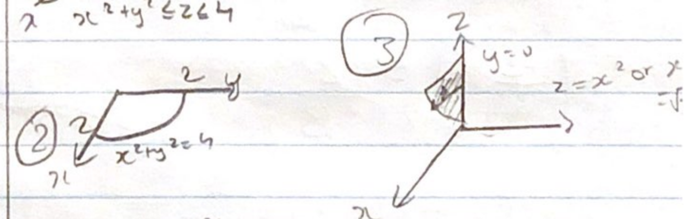
$$\int_0^1\left(\frac{xy^2}{2}\Big|_{x^2}^{x}\right)dx=\int_0^1 \frac{x^3}{2}-\frac{x^5}{2}\,dx$$

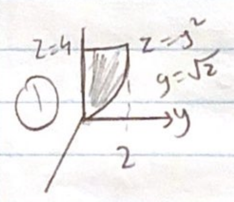
$$=\frac{x^4}{8}-\frac{x^6}{12}\Big|_0^1=\frac{1}{8}-\frac{1}{12}$$

Example 2 Evaluate $\iiint_e x\,dV$ where e is the solid in the first octant, enclosed by x $z=x^2+y^2$ and $z=4$



$x^2+y^2\le z\le 4$





$dV=\begin{cases} dx\,dy\,dz & dx\,dz\,dy \\ dz\,dy\,dx & dz\,dx\,dy \\ dy\,dx\,dz & dy\,dz\,dx\end{cases}$

① $dV=dx\,dy\,dz$

$R_{zy}=\begin{cases}0\le z\le 4\\ 0\le y\le \sqrt{z}\end{cases}$

$$\int_0^4\int_0^{\sqrt{z}}\int_0^{\sqrt{z-y^2}} x\,dx\,dy\,dz$$

$0\le x\le \sqrt{z-y^2}$

$z=x^2+y^2 \Rightarrow x^2=z-y^2 \Rightarrow x=\sqrt{z-y^2}$

(2) $dv = dzdydx$ $R_{xy} = \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4-x} \end{cases}$

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} x \, dz\,dy\,dx$$

(3) $dv = dy\,dx\,dz$

$0 \le z \le 4$
$0 \le x \le \sqrt{z}$
$0 \le y \le \sqrt{z-x^2}$
$y = \sqrt{z-x^2}$

$$\int_0^4 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} x\,dy\,dx\,dz$$

$$\int_0^4 \int_0^{\sqrt{z}} \left(xy \Big|_0^{\sqrt{z-x^2}} \right) dx\,dz$$

$$= \int_0^4 \int_0^{\sqrt{z}} x\sqrt{z-x^2}\,dx\,dz = \frac{1}{3}\int_0^4 z^{3/2}\,dz = \frac{1}{3}\frac{z^{5/2}}{5/2}\Bigg|_0^4$$

$$\int_0^{\sqrt{z}} x\sqrt{z-x^2}\,dx = \frac{-(z-x^2)}{2x^{3/2}}\Bigg|_0^{\sqrt{z}} = \frac{-(z-x^2)^{3/2}}{3}\Bigg|_0^{\sqrt{z}}$$

$$-\left(\frac{-z^{3/2}}{3}\right) = \frac{z^{3/2}}{3}$$

final answer $= \dfrac{32 \times 2}{3 \times 5}$

Change of variables in triple integrals

Double integrals

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$dA = J \times dA$

Triple integrals

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

Example 4. Find the jacobian of the change of coordinates to
1. spherical coordinates
2. cylindrical coordinates

$(x,y,z) \rightsquigarrow (\rho, \theta, \phi)$

$x = \rho\cos\theta\sin\theta$ $y = \rho\sin\theta\sin\theta$ $z = \rho\cos\phi$

$x_\rho = \cos\theta\sin\theta$ $x_\theta = -\rho\sin\theta\sin\phi$ $x_\phi = \rho\cos\theta\cos\phi$

$y_\rho = \sin\theta\sin\phi$ $y_\theta = \rho\cos\theta\sin\phi$ $y_\phi = \rho\sin\theta\cos\phi$

$z_\rho = \cos\phi$ $z_\theta = 0$ $z_\phi = -\rho\sin\phi$

$$J = \det \begin{pmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix}$$

$\cos\theta\sin\phi(-\rho\cos\theta\sin\phi\,\rho\sin\phi) - (-\rho\sin\theta\sin\phi)(-\rho\sin\phi\sin\theta\sin\phi - \rho\sin\theta\cos^2\phi)$

$+ (\rho\cos\theta\cos\phi)(-\cos\phi\,\rho\cos\theta\sin\phi)$

$= -\rho^2\cos^2\theta\sin\phi^2 - \rho^2\sin\phi^2\sin\theta^2 - \rho^2\sin\theta^2\sin\phi\cos^2\phi$

$-\rho^2\sin^3\phi - \rho^2\cos^2\phi\sin\phi \quad -\rho^2\cos\theta^2\cos\phi^2\sin\phi$

$= -\rho^2\left(\sin\phi\sin^2\phi + \sin\phi\cos^2\phi\right)$

$= -\rho^2\sin\phi$

$\rho^2\sin\phi\cos^2\phi \quad \sin^2\theta + \rho^2\sin\phi\cos^2\phi\cos^2\theta$

$\rho^2(\sin\phi\cos^2\phi)(\sin^2\theta+\cos^2\theta) = \rho^2\sin\phi\cos^2\phi$

$\partial \quad \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2\sin\phi \qquad \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$

$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a(ei-fh)-b(di-fg)+c(dh-eg)$

Example 5. Evaluate the triple integral

$$\int_{-1}^{1}\int_0^2\int_0^{\sqrt{4-z^2}} y^2 z\,dx\,dz\,dy$$

Method 1 $\displaystyle\int_{-1}^{1}\int_0^2 y^2 z\sqrt{4-z^2}\,dz\,dy$

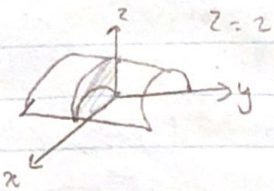
$$= \int_{-1}^{1} y^2 \int_0^2 z\sqrt{4-z^2}\,dz\,dy$$

$$\int_0^2 z\sqrt{4-z^2}\,dz = \frac{-(4-z^2)^{3/2}}{2\cdot 3/2}\Bigg|_0^2 = \frac{4^{3/2}}{3} = \frac{8}{3}$$

$$\int_{-1}^{1}\frac{8}{3}y^2\,dy = \frac{8}{3}\left(\frac{y^3}{3}\Bigg|_{-1}^{1}\right)$$

$= \dots$

Method 2



$0 \le x \le \sqrt{4-z^2}$

$x = \sqrt{4-z^2} \implies x^2+z^2 = 4$

$(r,\theta,y) \quad \begin{cases} x = r\cos\theta \\ z = r\sin\theta \\ y = y \end{cases}$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,y)} = r$$

$$\int_{-1}^{1}\int_0^{\pi}\int_0^2 y^2 r\sin\theta\, r\,dr\,d\theta\,dy$$

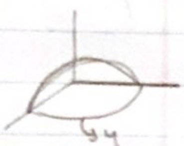
Lecture 21

Describing solids in spherical and cylindrical coordinates

Example 1 (Ex set 46, 11). Evaluate the integral $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} z\,dz\,dy\,dx$ by change of the order to $dx\,dz\,dy$

~~Example~~ $$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} z\,dz\,dy\,dx$$

want to change to $dx\,dz\,dy$



$0 \le y \le \sqrt{1-x^2}$

$y^2 = 1-x^2$

$y^2+x^2=1$

$x=\sqrt{1-y^2}$

$x = 1-y^2-z^2$

$x=\sqrt{1-y^2-z^2}$

$0 < y < \sqrt{1-x}$

$y=\sqrt{1-x}$

$0 < z \le \sqrt{1-x^2-y^2}$

$z^2 = 1-x^2-y^2$

$z\ \sqrt{1-y^2-z^2}$

x

$$\int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{\sqrt{1-y^2-z^2}} z\,dx\,dz\,dy$$

$$\int_0^{\sqrt{1-y^2-z^2}} z\,dx = xz\Big|_0^{\sqrt{1-z^2-y^2}} = z\sqrt{1-z^2-y^2}$$

$$\int_0^{\sqrt{1-y^2}} z\sqrt{1-z^2-y^2}\,dz = -\tfrac{1}{3}\left(1-z^2-y^2\right)^{3/2}\Big|_0^{\sqrt{1-y^2}}$$

→ detailed integration

$$\int z\sqrt{1-z^2-y^2}\,dz$$

$1-z^2-y^2 = u = -2z\,dz = du = z\,dz = \frac{du}{-2}$

$$= -\tfrac{1}{3}\int\sqrt{u}\,du = -\tfrac{1}{2}\,\frac{2u^{3/2}}{3} = -\tfrac{1}{3}u^{3/2}$$

$$= -\tfrac{1}{3}\left(1-z^2-y^2\right)^{3/2}\Big|_0^{\sqrt{1-y^2}} = -\tfrac{1}{3}\left(0-(1-y^2)^{3/2}\right)$$

$$= \tfrac{1}{3}(1-y^2)^{3/2}$$

next integration

$$\tfrac{1}{3}\int_0^1 (1-y^2)^{3/2}\,dy$$

↳ need to use trigonometric substitution

or use the calc

$$\int(1-y^2)^{3/2}\,dy = \quad y=\sin\theta \Rightarrow dy=\cos\theta\,d\theta$$

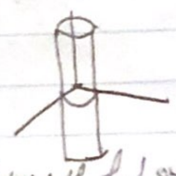
$$= \int(\cos^2)^{3/2}\cos\theta\,d\theta = \int\cos^4\theta\,d\theta$$

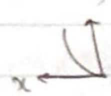
End of ex 1

Example 2: Find the volume of the solid above xy-plane and enclosed by the surfaces $x^2+y^2=1$ and $z=x^2-y^2$

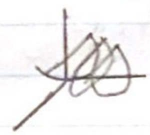
$z=x^2-y^2$

$x^2+y^2=1$





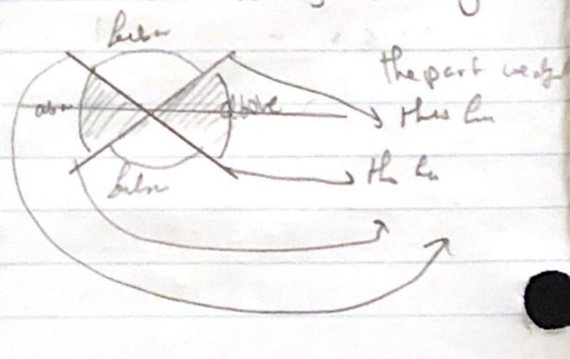
sketch the hard one first



→ check 3d plot online

$\boxed{\sqrt{1-y^2}}$ The intersection of the surface

$z=0 \Rightarrow x^2=y^2 \quad x=\pm y$



$$\int_R\int\int_0^{x^2=y^2} = \int_{-\pi/4}^{\pi/4}\int_0^1\int^{x^2=y^2} r\,dz\,dr\,d\theta$$

$R = R_1 \cup R_2$

$$2\int_0\int_{-x}^{x}\int_0^{x^2=y^2} d\,z\,dy\,dx$$

$$= 2\int_{-\pi/4}^{\pi/4}\int_0^1\int_0^{r^2\cos 2\theta} r\,dz\,dr\,d\theta$$

$$= 2\int_{-\pi/4}^{\pi/4}\int_0^1 r^3\cos 2\theta\,dr\,d\theta = 2\int_{-\pi/4}^{\pi/4}\tfrac{1}{4}\cos 2\theta$$

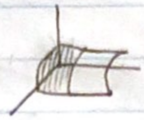
$$= \frac{r^4}{4}\Big|_0^1 = \tfrac{1}{4}$$

Example 3. Use cylindrical coordinates to evaluate the triple integral $\int_0^1\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} y^2 z\,dz\,dx\,dy$

$0 \le z \le \sqrt{4-x^2}$ $\qquad z = \sqrt{4-x^2}$

$-2 \le x \le 2$

$0 \le y \le 1$

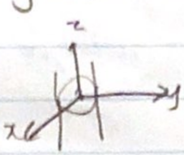


we need to use cylindrical coordinates

$z = z$

$x = r\cos\theta$

$y = r\sin\theta$

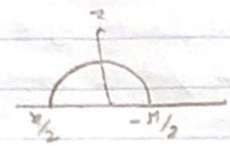


for this case we will do

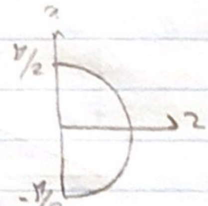
$y = y$

$z = r\cos\theta$

$x = r\sin\theta$



$$\int r\int_0^2\int_0^1 dy\,dr$$



you will have to use the form

$$\int_{-\pi/2}^{\pi/2}\int_0^2\int_0^1 r\,y^2\,r\cos\theta\,dy\,dr\,d\theta$$

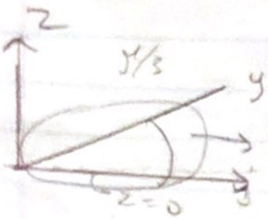
Jacobian

$$\tfrac{1}{3}\int_{-\pi/2}^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta$$

$$\int_0^2 r^2\,dr = \frac{r^3}{3}\Big|_0^2 = 8/3$$

$$8/3 \cdot 1/3 = 8/9\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta = 16/9$$

Example 4. Sketch the region R in y-z plane bounded by $z=0$, $y=\sqrt{3}z$ and $y^2+z^2=1$ then find the volume of the solid given by rotation of R about z-axis

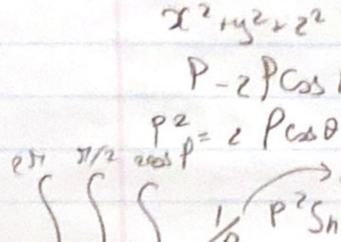


$$\iiint_T dV_{xyz}$$

$$= \int_0^{2\pi}\int_0^{1}\int_{\pi/3}^{\pi/2} \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta$$

Example 5: Evaluate $\iiint_T \frac{1}{\sqrt{x^2+y^2+z^2}} \, dV$,

where T is the solid in the unit sphere right above the origin



$x^2+y^2+(z-1)^2=1$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi/2$

$0 \le \rho \le 2\cos\theta$

$x^2+y^2+z^2-2z+1 = 1$

$\rho - 2\rho\cos\theta = 0$

$\rho^2 = 2\rho\cos\theta \Rightarrow \rho = 2\cos\theta$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\rho} \frac{1}{\rho} \, \rho^2 \sin\varphi \, d\varphi \, d\phi \, d\theta$$

Jacobian

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Lecture 22

Divergence Theorem

Divergence theorem involves triple integrals and surface integrals.

Let $\mathcal{E}$ be a closed solid region in $\mathbb{R}^3$ with the surface boundary $\mathcal{C}$ and let n be the outward normal vector to $\mathcal{C}$. If $F = (P, Q, R)$ is a nice vector field defined over $\mathcal{E}$

$$\iiint_{\mathcal{E}} (P_x + Q_y + R_z)\, dV = \iiint_{\mathcal{E}} \operatorname{div} F\, dV$$

$$= \iint_{\mathcal{C}} F \cdot dS = \iint_{\mathcal{C}} F \cdot n\, dS$$

Example 1. Let $F(x,y,z) = (2x, 3y, 4z)$ and let $\mathcal{C}$ be the sphere centred at the origin with radius 2

$$F(x,y,z) = (2x, 3y, 4z)$$

$$\iint_S F\, d\mathbf{S} \to \text{bold } S$$

$$\operatorname{div} F = (2x)_x + (3y)_y + (4z)_z = 9$$

$$\iiint_{\mathcal{E}} 9\, dV = 9 \iiint_T dV$$

$$= 9 \times \tfrac{4}{3}\pi (2)^2$$

volume of sphere $= \tfrac{4}{3}\pi r^2$.

Example 2

Let $\mathcal{E}$ be a closed solid region in $\mathbb{R}^3$ with the surface boundary $\mathcal{C}$. The flux of any of the vector fields $(x,0,0)$, $(0,y,0)$, and $(0,0,z)$ over $\mathcal{C}$ gives the volume $\mathcal{E}$

$$F(x,y,z) = (0, y, 0)$$

$$\operatorname{div} F = 0 + 1 + 0 = 1$$

$$\iint_S (0,y,0)\, d\mathbf{S} = \iiint_{\mathcal{E}} 1\, dV = \text{volume of solid}$$

Example 3 (Ex set 48-9) Show that the volume of a solid region $\mathcal{E}$ enclosed by a surface $\mathcal{C}$ is given by

$$V = \tfrac{1}{6} \iint_{\mathcal{C}} (\nabla |r|^2)\, d\mathbf{S}$$

where $r = (x,y,z)$ is the position vector

$$r = (x,y,z) = |r|^2 = x^2 + y^2 + z^2$$

$$F = \nabla(|r|^2) = (2x, 2y, 2z)$$

$$\operatorname{div} F = 2 + 2 + 2 = 6$$

divergence theorem $\tfrac{1}{6} \iiint 6\, dV = V$

Example 4 ~~Exer~~ Let $f(x,y,z) = x^4+y^4+z^4$
Calculate the flux of ∇f over the sphere whose center is at the origin having radius of 3

$$f(x,y,z) = x^4+y^4+z^4$$

$$\iint_S \nabla f \, dS$$

$S \to$ the sphere at origin with $r=3$

$$\nabla f = 4x^3, 4y^3, 4z^3$$

Jacobian

$$= 12\iiint_E x^2+y^2+z^2 \, dV = 12\int_0^{2\pi}\int_0^{\pi}\int_0^3 \rho^2\rho^2 \sin\phi \boxed{d\rho\, d\phi\, d\theta}$$

$$\int_0^3 \rho^4 \, d\rho = \frac{\rho^5}{5}\Big|_0^3$$

$$12\int_0^{2\pi}\int_0^{\pi} \frac{3^5}{5} \sin\phi \, d\phi \, d\theta$$

$$\int_0^{\pi} \sin\phi \, d\phi = -\cos\phi \Big|_0^{\pi} = 1+1 = 2$$

$$\frac{12 \times 2 \times 3^5}{5}\int_0^{2\pi} d\theta$$

Maxima and Minima

Let F define a surface τ given by $z=f(x,y)$.
In order to find max/min of the function f. we need to look for points $P=(x_0, y_0)$ for which both following equations hold
Such points are critical points.

$$\frac{\partial f}{\partial x} = f_x = z_x = 0 \quad \frac{\partial f}{\partial y} = f_y = z_y = 0$$

But note that no necessarily any point satisfying both these equations is a Max/Min point in order to find max/min points we need to evaluate the following equation at the critical points

$$D = z_{xx} \cdot z_{yy} - z_{xy}^2$$

D is called the descriminant

at a point $P = (x_0, y_0)$

1. If $D<0$, then P is neither max or min. It's saddle point
2. If $D>0$ and $z_{xx}>0$, then P is relative minimum point
3. If $D>0$ and $z_{xx}<0$ then P is a relative maximum point
4. If $D=0$, anything can happen

Example 5

for all the following functions, classify all local max and minimum ~~points~~

① $f(x,y) = x^2 + y^2 - 2x + 4y + 6$

critical points

$f_x = 2x - 2 = 0$ at $x = 1$

$f_y = 2y + 4 = 0$ at $y = -2$

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$= 2 \times 2 - 0 = 4$ $\quad 4 > 0$

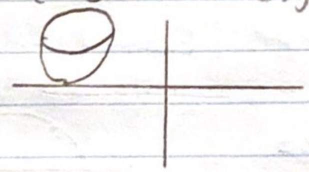
$f_{xx} = 2 > 0 \Rightarrow (1,-2)$ is a local minimum

$f(x,y) = x^2 + y^2 - 2x + 4y + 6 = (x-1)^2 + (y+2)^2 + 1$

$z = (x-1)^2 + (y+2)^2$

$= x^2 + y^2$ is a paraboloid

bottom point at $(x,y) = (0,0)$ or at $(x,y) = (1,-2)$



~~② $f(x,y) = x^2 + y^2 - 2x + 4y + 6$~~

② $f(x,y) = (x+y-1)^2$

$= x^2 + y^2 + 2xy - 2x - 2y + 1$

$$\left.\begin{aligned} f_x &= 2x + 2y - 2 = 0 \\ f_y &= 2y + 2x - 2 = 0 \end{aligned}\right\} \rightarrow x^2 + y^2 = 1$$

all the points at this line are critical

$f(x,y) = (x+y-1)^2 = ((x+y)-1)^2$

$= (x+y)^2 - 2(x+y) + 1 =$

$X^2 - 2X + 1$

$x - y = 1$ $\quad (x-1)^2 \rightarrow$ parabola

therefore it is a cylinder because one of the coordinates are missing

Lecture 22

Lecture 23

Maxima and Minima

Example 1. Find the largest value of $f(x,y,z) = xyz^2$ when (x,y,z) are on the plane $x+2y+3z=24$

$x+2y+3z=24$

$x = 24-2y-3z$

$f(y,z) = (24-2y-3z)yz^2$ } → find min & max

① find the critical points

$f(y,z) = (24-2y-3z)yz^2 = 24yz^2 - 2y^2z^2 - 3yz^3$

$f_y = 24z^2 - 4yz^2 - 3z^3 = 0 \Rightarrow z^2(24-4y-3z)=0$

$f_z = 48yz - 4y^2z - 9yz^2 = 0 \Rightarrow z(48-4y^2-9yz)=0$

① $z=0$

$24-4y-3z=0$

② $z=0$

$y(48-4y-9z)=0$

if $z=0 \longrightarrow f_y = f_z = 0$ $(y,0)$ is a critical point

if $24-4y-3z=0 \Rightarrow 3z=0$

$z = \frac{24-4y}{3} = 8 - \frac{4}{3}z$

$f_z = 0 \Rightarrow y(48-4y-9(8-\frac{4}{3})y) = 0$ $f_y = 0$

$\Rightarrow \begin{cases} y=0 \\ -24+8y=0 \end{cases} \Rightarrow \begin{cases} y=0 \\ \text{or} \\ y=3 \end{cases}$

wrap it all

$z=0$ for any

$y=0$ & $z=8$

$y=3$ & $z=4$

$D(y,z) = f_{yy} \cdot f_{zz} - (f_{yz})^2$

$f_y = 24z^2 - 4yz^2 - 3z^3$

$f_z = 48yz - 4y^2z - 9yz^2$

$x=0 = f_{yy} = 0$

$f_{yz} = 0 \Rightarrow D = 0$

$y=0, z=8$

$f_{yy} = -4(64)$ $f_{zz} = 0$

$f_{yz} = 48(8) - 9(64) < 0$

$\Rightarrow D<0$ & Saddle point

$y=3$ $z=4$

$D = (-4(4)^2)(48(3) - 4(3+18\times 12$

$-(48(4) - 8(3)(4) - 9\times 16)^2$

if $D<0 \Rightarrow$ Saddle point

if $D>0 \Rightarrow$ Since $f_{yy}<0$ Maximum

Example 2. find the Area of the largest rectangle sitting in a semi-circle of radius a

$2xy$ r.t $x^2+y^2=a^2$

$x^2+y^2=a^2$

$y = \sqrt{a^2-x^2}$

max of $2x(\sqrt{a^2-x^2})$

you don't need to think of

max of $f(x) = 2x\sqrt{a^2-x^2}$

$f'x = 2\sqrt{a^2-x^2} = \frac{-2x^2}{\sqrt{a^2-x^2}}$

$= \frac{2a^2-2x^2-2x^2}{\sqrt{a^2-x^2}} = 0$

$x = \frac{a}{\sqrt{2}}$

Lagrange Multiples

In this section we would like to optimize a function $f(x,y,z)$ respect to some constraint $g(x,y,z)=0$. Then how to do so,

Create a function $\mathcal{L}(x,y,z,\lambda)$ with a new variable λ defined as

$$\mathcal{L}(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z)$$

Then we find the critical points $\mathcal{L}$ namely we solve

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial z} = 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

Note: the same argument can be said about 2 variables functions $f(x,y)$

And finally find the Max/Min of $\mathcal{L}$

Example 3 : find the area of the largest rectangle sitting in a semicircle of radius a

$g(x,y) = x^2 - y + 1$

max $f(x,y) = x^2 + y^2$

$g(x,y) = 0 \qquad y = x^2 - 1$

the point of Maximum for slope of $f(\mathcal{L})$

$- \int \nabla f = \nabla \mathcal{L}(x,y,z,\lambda) = \lambda g$

$= -g(x,y,z) = 0 \quad \lambda_x = 0 \Rightarrow f_x = \lambda g_x$

$f_y = 0 \Rightarrow f_y = f g_y$

$f_z = 0 \Rightarrow f_z = \lambda g_z$

max $2xy$ such that $x^2 + y^2 = a^2$

$g(x,y) = x^2 + y^2 = a^2 = 0$

$g(x,y) = x^2 + y^2 - a^2$

$f(x,y,\lambda) = 2xy - \lambda(x^2 + y^2 - a^2)$

$y = \lambda(\lambda y) = \lambda^2 y$

$\mathcal{L}_x = 2y = 2\lambda x = 0 \Rightarrow y = \lambda x$

$\mathcal{L}_y = 2x - 2\lambda y = 0 \Rightarrow x = \lambda y$

$\mathcal{L}_\lambda = x^2 + y^2 - a^2 = 0$

$\Rightarrow \int \lambda^2 = 1$

if $y = 0 \Rightarrow x = a \qquad y = 0$

$\Rightarrow \lambda^2 = 1 \quad \lambda = \pm 1 \quad \Rightarrow y = \pm x$

$\Rightarrow x^2 + (\pm x)^2 = a^2 \Rightarrow 2x^2 = a^2$

$\Rightarrow x = a/\sqrt{2}$

Example 4 Maximize the function $x^2 + xy - 3y^2$ subject to the constraint $x + 2y = 2$

$\mathcal{L}(x,y,\lambda) = x^2 + xy - 3y^2 - \lambda(x + 2y - 2)$

$= \mathcal{L}_x = 2x + y - \lambda \Rightarrow y = \lambda - 2x$

$= \mathcal{L}_y = x - 6y - 2\lambda \Rightarrow y = \frac{x - 2\lambda}{6}$

$\lambda - 2x = \frac{x - 2\lambda}{6}$

$6\lambda - 12x = x - 2\lambda$

$8\lambda = 13x = \lambda$

$= \mathcal{L}_\lambda : \to$ solve for λ

$6\lambda - 12x = x - 2\lambda$

$8\lambda = 13x \Rightarrow x = \frac{8}{13}\lambda \Rightarrow y = \frac{8/13\,\lambda - 2\lambda}{6} = \frac{-18}{13}$

$= -3/13\,\lambda$

from $\underline{y = \frac{x - 2\lambda}{6}}$

$x + 2y - 2 = 0 = \frac{8}{13}\lambda + 2 : \frac{-3}{13}\lambda - 2 = 0$

$\Rightarrow \frac{2}{13}\lambda = 2 \Rightarrow \lambda = 13$

$\lambda = 13 \Rightarrow \left.\begin{matrix} x = 8 \\ y = -3 \end{matrix}\right\}$

either the point

a max or min occur so we need to check other point

$f(8,-3) = 64 - 24 - 27 = 13$

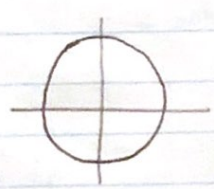
$g(2,0) = 0$

So the point satisfy the condition

So this is Max.

Example 5 find the Maximum and Minimum value of
$f(x,y) = 3x^2 + 2y^2 - 2x - 1$ for $(x,y) \in R = \{x,y)$
so that $x^2 + y^2 \leq 9\}$

$f(x,y) = 3x^2 + 2y^2 - 2 - 1$
$g(x,y) = x^2 + y^2 - 9 \leq 0$



we need to find the critical point

$f_x = 0 \rightarrow$ doesn't exist
$f_y = 0 \rightarrow$ doesn't exist

So max of f
at $g = 0$