CARLETON UNIVERSITY

FINAL EXAMINATION
MATH 2004
SOLUTIONS
Fall 2016

DURATION: 3 HOURS

Department Name and Course Number: School of Mathematics and Statistics, MATH 2004
Course Instructor(s): Dr. A.B. Mingarelli (Sect. A), Dr. X. Hua (Sect. B, C), Dr. R. Cova (Sect. D)

AUTHORIZED MEMORANDA
SCIENTIFIC CALCULATOR ONLY AS PER COURSE OUTLINE.

In addition to the examination paper students will require an EXAMINATION BOOKLET, and a SCANTRON SHEET. This exam may be released to the Library.

1. Please verify that you are in possession of a SCANTRON FORM
2. Please fill in your COURSE CODE (e.g., MATH 2004) and COURSE SECTION (e.g., A, B, C, D), as per above list of instructors; YOUR NAME and YOUR STUDENT NUMBER where required on the Scantron form AND on this examination.
3. **The entire examination consists of 8 pages and two parts, A and B, and is marked out of a total of 80**, that is, 40% of your final mark. Part A consists of 12 multiple choice questions each worth 3 marks for a total of 36 marks. **Please fill in only one answer on your Scantron sheets with a pencil** as there is only one answer to any given question. Circling two or more answers to any question invalidates that question (*i.e.,* you get 0 marks for that question). Part B consists of 6 questions (with details required), for a total of 44 marks for that section. **Both part a and b must be submitted along with the scantron sheet, so do not detach nor unstaple this examination. DO NOT SUBMIT THE EXAMINATION BOOKLET CONTAINING ROUGH WORK**.

Print Name :

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Section (either A, B, C, or D. See above for your Instructor's name):

PART A

Do all (12) Questions for a total of 36 marks out of a maximum of 80.
Do not detach nor unstaple this examination. Missing sheets will void any credit for those questions.

- **A1.** The straight line $2x + 3y = 1$ has a normal vector equal to

  (a) $(3, 2)$ (b) $(2, 3)$ (c) $(3, -2)$ (d) $(2, -3)$ (e) None of these

  Solution: (b)

- **A2.** The plane containing the point $(1, 1, 1)$ and whose normal vector is given by $(2, 0, -1)$ is given by

  (a) $x + y - z = 1$ (b) $x - 2z = -1$ (c) $2y - z = 1$ (d) $2x - z = 1$ (e) None of these

  Solution: (d)

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- **A3.** Find the area of the triangle ΔOPQ whose vertices are the points $< 0, 0, 0 >$, $< 1, 2, 0 >$ and $< -2, 1, 0 >$. (**Note:** In this examination the symbols (a, b, c) and $< a, b, c >$ both denote the vector with components a, b, c, etc.)

  (a) $\dfrac{\sqrt{21}}{2}$ (b) $\sqrt{21}$ (c) $\dfrac{\sqrt{15}}{2}$ (d) 2 (e) None of these

  Solution: (a)

- **A4.** Find the curl of the vector field $\mathbf{F}(x, y, z) = x^2y\,\mathbf{i} - xz\,\mathbf{j} + 2yz\,\mathbf{k}$ at the point $(1, 0, -1)$.

  (a) $(1, 0, 0)$ (b) $(0, 1, 0)$ (c) $(0, 0, -1)$ (d) $(-1, 0, 0)$ (e) None of these

  Solution: (d)

- **A5.** Let $f(x, y)$ where $x = u(s, t), y = v(s, t)$ all be differentiable functions. At the point $(s, t) = (0, 1)$ let $f_x = 1$, $f_y = 2$, $x_t = -1$ and $y_t = 1$. Evaluate $\dfrac{\partial f}{\partial t}$ when $s = 0$, $t = 1$.

  (a) 1 (b) -1 (c) 2 (d) -2 (e) None of these

  Solution: (a)

- **A6.** Which one of the following vector fields on the plane is conservative?

  (a) $< x\cos y, y\cos x >$ (b) $< 2x + y, x >$ (c) $< x^2y - 1, xy^2 + 1 >$ (d) $< 2y + x, x >$ (e) None of these

  Solution: (b)

- **A7.** Find the area of the region enclosed by the curves $\theta = 0$, $\theta = \pi/3$ and $r = 2\sec\theta$.

  a) $\sqrt{3}$ (b) $\dfrac{8\sqrt{3}}{5}$ (c) $2\sqrt{3}$ (d) $\dfrac{\sqrt{3}}{2}$ (e) None of these

  Solution: (c)

- **A8.** Let f be continuous in the region R given by $R = \{(x, y) : 2x \le y \le x^2 + 1,\ 0 \le x \le 1\}$. Which of the following iterated integrals is equal to $\displaystyle\int_0^1 \int_{2x}^{x^2+1} f(x, y)\,dy\,dx$.

  (a) $\displaystyle\int_0^2 \int_0^{y/2} f(x, y)\,dx\,dy$ (b) $\displaystyle\int_0^2 \int_{\sqrt{y-1}}^{y/2} f(x, y)\,dx\,dy$

  (c) $\displaystyle\int_0^1 \int_0^{\sqrt{y-1}} f(x, y)\,dx\,dy + \int_1^2 \int_0^{y/2} f(x, y)\,dx\,dy$ (d) $\displaystyle\int_0^1 \int_0^{y/2} f(x, y)\,dx\,dy + \int_1^2 \int_{\sqrt{y-1}}^{y/2} f(x, y)\,dx\,dy$

  (e) None of these

  Solution: (d)

- **A9.** Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + xy + y^2}$.

  (a) 0 (b) $\dfrac{1}{2}$ (c) -2 (d) The limit does not exist (e) None of these

  Solution: (d)

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- **A10**. Evaluate the triple integral $\displaystyle\int_{-5}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx.$

  (a) $\frac{25\pi}{2}$ (b) $\frac{\pi}{2}$ (c) 25π (d) $\frac{25\pi}{4}$ (e) None of these

  Solution: (a)

- **A11.** The parametrization $x = \frac{av}{c}\cos u,\ y = \frac{bv}{c}\sin u,\ z = \frac{v}{c}$, where a, b, c are constants $(a \neq 0, b \neq 0, c \neq 0)$, and $u \in [0, 2\pi], v \in \mathbf{R}$, describes which of the following surfaces?

  (a) A hyperbolic cylinder (b) An elliptic cylinder (c) An elliptic cone (d) A torus (e) None of these

  Solution: (c)

- **A12.** Evaluate the line integral with respect to arc length, $\displaystyle\int_C xyz\, ds$ where C is the line segment joining $(0, 0, 0)$ to $(1, 1, 1)$.

  (a) $\frac{\sqrt{3}}{3}$ (b) $\frac{\sqrt{3}}{4}$ (c) $\frac{1}{4}$ (d) $\frac{1}{3}$ (e) None of these

  Solution: (b)

End of Part A

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PART B

Do All six (6) Questions for a total of 44 marks out of a maximum of 80.
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- **B1.** [7 marks] Evaluate $I = \int_C (y^2 - 1)\,dx + (x^2 + 1)\,dy$ where C is the perimeter of the square whose vertices are $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ with c.c. orientation.

Solution: Use Green's Theorem. Here $P = y^2 - 1$, $Q = x^2 + 1$ so that $Q_x - P_y = 2(x - y)$. It follows that

$$\begin{aligned} I &= \int_0^1 \int_0^1 (2x - 2y)\,dy\,dx \\ &= 0. \end{aligned}$$

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- **B2.** [2+5 marks]

1. Let $\mathbf{F}(x, y, z) = (y, x + z, y)$. Show that $\mathbf{F}$ is a conservative vector field.
2. Let $f(1, -1, 0) = -1$ where f is that scalar field such that $\nabla f = \mathbf{F}$. Calculate $f(1, 1, 2)$.

Solution: a) $\nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & x+z & y \end{pmatrix} = \mathbf{0}.$

b) Use the key result that

$$f(x, y, z) = \int_C \mathbf{F} \cdot d\mathbf{r} + c,$$

where c is a constant and C is line segment joining any point, say, $P_0(0, 0, 0)$ to $P(x, y, z)$. (See Example 224, p. 301, in the text.)

Here $\mathbf{r}(t) = (tx, ty, tz)$, $t \in [0, 1]$ is a parametrization of the line segment joining the two points in question so that

$$\begin{aligned} f(x, y, z) &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^1 \mathbf{F}(tx, ty, tz) \cdot (x, y, z)\, dt \\ &= \int_0^1 (ty, tx + tz, ty) \cdot (x, y, z)\, dt \\ &= \int_0^1 (2txy + 2tyz)\, dt \\ &= xyt^2 \Big|_{t=0}^{1} + yzt^2 \Big|_{t=0}^{1} \\ &= xy + yz + c. \end{aligned}$$

Since $f(1, -1, 0) = -1$ it follows that $c = 0$. In turn, this means that $f(1, 1, 2) = 1 + 2 = 3$.

- **B3.** [7 marks] Evaluate $I = \iint_{\mathcal{R}} \sin(9x^2 + 4y^2)\, dA$ where R is the region in first quadrant enclosed by the ellipse $9x^2 + 4y^2 = 1$ and the lines $x = 0$ and $y = 0$.

Solution: Use the elliptic coordinate transformation $3x = r\cos\theta$, $2y = r\sin\theta$ whose Jacobian is $r/6$. Then this change of variables converts I into

$$\begin{aligned} I &= \frac{1}{6} \int_0^{\pi/2} \int_0^1 r \sin r^2\, dr\, d\theta \\ &= \frac{\pi}{12} \int_0^1 r \sin r^2\, dr \\ &= \frac{\pi}{24} (1 - \cos 1). \end{aligned}$$

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- **B4.** [10 marks] Let $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ be a vector field and let S be that part of the plane $y + z = 2$ cut by the cylinder $x^2 + y^2 = 1$. Find the flux of $\nabla \times \mathbf{F}$ through the surface S.

Solution: We need to evaluate the surface integral $I = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ where $\mathbf{n}$ is an outward unit normal to S. A simple calculation gives $\nabla \times \mathbf{F} = (0, 0, 1 + 2y)$. Projecting S onto the xy-plane we get that its projection R is given by the disk $x^2 + y^2 \leq 1$. On the other hand, the surface S is of the form $z = f(x, y) = 2 - y$ and, since it is a plane, its outward normal is $(0, 1, 1)$, so $\mathbf{n} = (0, 1, 1)/\sqrt{2}$. It follows that

$$
\begin{aligned}
I &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS \\
&= \frac{1}{\sqrt{2}} \iint_R (0, 0, 1 + 2y) \cdot (0, 1, 1)\, dS \\
&= \frac{1}{\sqrt{2}} \iint_R (1 + 2y)\, \sqrt{1 + 0^2 + (-1)^2}\, dA \\
&= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta)\, r\, dr\, d\theta \\
&= \pi.
\end{aligned}
$$

Alternately, using Stokes' Theorem we know that the flux $I = \int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the perimeter of the unit circle on the xy-plane described in the positive (counterclockwise) sense. Thus, parametrizing C by $\mathbf{r}(t) = (\cos t, \sin t, 0)$ for $t \in [0, 2\pi]$ we get

$$
\begin{aligned}
I &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\
&= \int_0^{2\pi} (-\sin^2 t, \cos t, 0) \cdot (-\sin t, \cos t, 0)\, dt \\
&= \int_0^{2\pi} (\sin^3 t + \cos^2 t)\, dt \\
&= \int_0^{2\pi} \left((1 - \cos^2 t) \sin t + \frac{1}{2}(1 + \cos 2t) \right) dt \\
&= \ldots \\
&= \pi.
\end{aligned}
$$

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- **B5.** [7 marks] Find the volume of the tetrahedral region T in the first octant enclosed by the coordinate planes and the plane $x + 2y + 3z = 6$.

Solution: Projecting T onto the xy-plane we need to evaluate the triple integral

$$V = \int_0^6 \int_0^{(6-x)/2} \int_0^{(6-x-2y)/3} dz\, dy\, dx = \frac{1}{3} \int_0^6 \int_0^{(6-x)/2} (6 - x - 2y)\, dy\, dx = 6,$$

or any one of the five other equivalent integrals.

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- **B6.** [6 marks] Find the flux of the vector field $\mathbf{F}(x, y, z) = 3x\,\mathbf{i} + y^2\,\mathbf{j} + xz\,\mathbf{k}$ over the rectangular box $T : 0 \le x \le 1$, $0 \le y \le 3$, $0 \le z \le 2$.

Solution: Use the Divergence Theorem. Thus,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_T \operatorname{div}\mathbf{F}\,dV.$$

But,

$$\begin{aligned}\iiint_T \operatorname{div}\mathbf{F}\,dV &= \iiint_T (3 + 2y + x)\,dV \\ &= \int_0^1\int_0^3\int_0^2 (3 + 2y + x)\,dz\,dy\,dx \\ &= 39.\end{aligned}$$