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Carleton UNIVERSITY

Final
EXAMINATION
December 2017

DURATION: 3 HOURS

No. of Students:

Department Name & Course Number: **Mathematics & Statistics MATH2004A,B,C,D,F**
Course Instructor(s) **X Hua (A, D), M. J. Moore (B), M. Lemire (C), J. Abdulrahman (F).**

AUTHORIZED MEMORANDA
NON-PROGRAMMABLE, NON-GRAPHING CALCULATORS PERMITTED

Students MUST count the number of pages in this examination question paper before beginning to write, and report any discrepancy to a proctor. This question paper has 6 pages.

This examination question paper MAY NOT be taken from the examination room.

In addition to this question paper, students require:

an examination booklet		no
a Scantron sheet	Yes	
blank paper for rough work	Yes	

COURSE INSTRUCTOR AND SECTION (See above) ____________

LAST NAME ____________ GIVEN NAME ____________

STUDENT # ____________ SIGNATURE ____________

ANSWER ALL QUESTIONS

In part A use a pencil to indicate your answer on the Scantron sheet.

In part B you must justify all of your assertions and show all of your work.

Only Scantron sheets and work on this booklet will be marked but you must submit your scratch paper also.
If you need additional space to write your answers, continue on the last page of this paper, noting "continued on page 6".

Questions	Maximum	Score
A	48	
B1	10	
B2	8	
B3	9	
B4	9	
B5	6	
B6	10	
Total	100	

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Multiple choice questions: use a pencil to select your answer on the Scantron sheet. **Only select one answer!**

A4 A12 ??

A1. Let θ be the angle between the two planes $x + 2y - 2z - 3 = 0$ and $2x + 2y - z - 5 = 0$. Find cos θ.

(a) $\frac{4}{9}$ (b) $\frac{9}{4}$ (c) $\frac{8}{9}$ (d) $\frac{9}{8}$ (e) $-\frac{4}{9}$

$\cos\theta = \dfrac{(1,2,-2)\cdot(2,2,-1)}{|v_1||v_2| = 9} = \boxed{\frac{8}{9}}$

A2. Find the length of the polar curve $r = \cos\theta + \sin\theta$, $0 \le \theta \le \pi/4$.

(a) $\sqrt{2}\pi/4$ (b) $\sqrt{2}\pi/2$ (c) $3\sqrt{2}\pi/4$ (d) $5\sqrt{2}\pi/4$ (e) $\pi/2$.

$l(c) = \int_\alpha^\beta \sqrt{\rho'^2(\theta) + \rho^2\theta}\, d\theta$

A3. Find the directional derivative of $f(x, y, z) = z^3 + e^{xy-1}$ at the point (1, 1, 2) in the direction of the vector $\underline{v} = (1, 2, 2)$.

(a) $2e/3$ (b) $5e/3$ (c) 4 (d) 5 (e) 9

$\dfrac{\nabla f \cdot \vec{v}}{|v|} = \dfrac{(f_x, f_y, f_z)\cdot \vec{v}}{|v|}$

A4. Find the tangent plane to the surface $x^3y - y^3x + z^3x = -5$ at the point P(1, 2, 1).

(a) $x+2y+z=6$ (b) $-7x+11y+5z=-20$ (c) $7x-11y-5z=-20$ (d) $7x-13y-11z=-30$ (e) $7x+13y+11z=44$

A5. The volume under the surface $z = 8 - 2x^2 - y^2$ which projects onto the region $\Re = \{(x, y) : 0 \le x \le 1, 0 \le y \le 2\}$ is

$\int_0^1\int_0^2 8 - 2x^2 - y^2\, dy\, dx$

(a) 8 (b) 5 (c) 12 (d) 4 (e) 1

A6. If the vectors $\underline{v} = 2\underline{i} - \underline{j} + \underline{k}$ and $\underline{w} = 3\underline{i} + 2t\underline{j} + 4\underline{k}$ are orthogonal, then t =

$= (2, -1, 1)\cdot(3, 2t, 4) = 0$

$6 + 4 = 2t \Rightarrow t = \boxed{5}$

(a) 8 (b) -1 (c) 0 (d) 5 (e) 13

A7. Let $w = xyz + x^2 + y^3 + z^4$ and $x = 2s - 3t$, $y = st^2$, $z = s^2 + t^3 - 1$, then $\left.\dfrac{\partial w}{\partial t}\right|_{s=1,t=1} =$

(a) 16 (b) 30 (c) 24 (d) 3 (e) 18

A8. Find curl **F** for the vector field $\mathbf{F}(x, y, z) = (3x^2, 2z, -x) = 3x^2\underline{i} + 2z\underline{j} - x\underline{k}$

(a) $5\underline{i} - 4\underline{j} + \underline{k}$ (b) $3\underline{i} - 2\underline{j} + \underline{k}$ (c) $-2\underline{i} + \underline{j}$ (d) $6\underline{k}$ (e) none of these

A9. Given the point $(\sqrt{2},0,1)$ in Cartesian coordinates, its cylindrical coordinates are

(a) $(\sqrt{2},0,1)$ (b) $(\sqrt{2},\pi,1)$ (c) $(\sqrt{3},0,1)$ (d) $(\sqrt{2},\pi/2,1)$ (e) none of these

A10. Given the point (-1,0,1) in Cartesian coordinates, its spherical coordinates are

(a) $(1,0,\pi/4)$ (b) $(\sqrt{2},0,\pi/4)$ (c) $(\sqrt{2},\pi/4,0)$ (d) $(\sqrt{2},\pi,\pi/4)$ (e) none of these

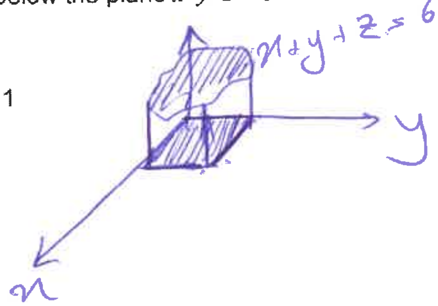
A11. The function $f(x, y) = x^2y - x^2 - 2y^2$ has critical points (0,0) and (2,1). These are, respectively:

(a) both maxima (b) both minima (c) a maximum and a minimum
(d) a minimum and a saddle point (e) a maximum and a saddle point

A12. Let E be the region in $\mathbb{R}^3$ above the square $[0,1]\times[0,1]$ and below the plane $x+y+z = 6$

$\iiint_E 6xy\,dV =$

(a) 3 (b) 5 (c) 2 (d) 4 (e) 1



$\int_0^1\int_0^1\int_0^{6-x-y} 6xy\, dz\, dx\, dy$

$6\int_0^1\int_0^1 xy(6-x-y-0)\, dx\, dy$

A2 $l(C) = \int_0^{\pi/4} \sqrt{(-\sin\theta + \cos\theta)^2 + (\cos\theta + \sin\theta)^2}\, d\theta$

$\rho(\theta) = \cos\theta + \sin\theta$

$\rho'(\theta) = -\sin\theta + \cos\theta$

$= \int_0^{\pi/4} \sqrt{2\sin^2\theta + 2\cos^2\theta}\, d\theta$

$= \int_0^{\frac{\pi}{4}} \sqrt{2}\, d\theta = \boxed{\sqrt{2} \times \pi/4}$

A3 $f_x = y e^{xy-1}$ $\quad f_x(1,1,2) = 1$

$f_y = x e^{xy-1}$ $\quad f_y(1,1,2) = 1$

$f_z = 3z^2$ $\quad f_z(1,1,2) = 12$

$\frac{(1,1,12) \cdot (1,2,2)}{|v|} = \frac{1+2+24}{3} = \frac{27}{3} = \boxed{9}$

A4 $x^3y - y^3x + z^3x = -5$ $\quad P(1,2,1)$

$\nabla f = (f_x, f_y, f_z)$

$f_x = 3x^2y - y^3 + z^3 \Rightarrow f_x(1,2,1) = -1$

$f_y = x^3 - 3y^2x \Rightarrow f_y(1,2,1) = -11$

$f_z = 3z^2x \Rightarrow f_z(1,2,1) = 3$

A4 Continue-

$\vec{n} = (-1, -11, 3)$

$-1x - 11y + 3z = d$ and the find d

$-1(x-1) - 11(y-2) + 3(z-1) = 0$

the exact same things.

$-x + 1 - 11y + 22 + 3z - 3 = 0$

A5 $\displaystyle\int_0^1 \int_0^2 8 - 2x^2 - y^2 \, dy \, dx = \int_0^1 \left(8y - 2x^2 y - \frac{y^3}{3} \Big|_0^2 \right) dx$

$\displaystyle = \int_0^1 16 - 4x^2 - \frac{8}{3} \, dx = 16x - \frac{4x^3}{3} - \frac{8}{3}x \Big|_0^1$

$\displaystyle = 16 - \frac{4}{3} - \frac{8}{3} = 16 - \frac{12}{3} = \frac{36}{3} = \boxed{12}$

~~A7~~ A7 $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{t} + \dfrac{\partial w}{\partial y}\cdot\dfrac{\partial y}{\partial t} + \dfrac{\partial w}{\partial z}\cdot\dfrac{\partial z}{\partial t}$

$w_x = yz + 2x,\ x_t = -3$

$w_y = xz + 3y^2,\quad y_t = 2st$

$w_z = xy + 4z^3,\quad z_t = 3t^2$

$s=1$ $\quad x(1,1) = -1 \quad y(1,1) = 1 \quad z(1,1) = 1$

$t=1$

$x_t = -3 \quad y_t(1,1) = 2 \quad z_t(1,1) = 3$

$w_x = yz + 2x = -1$

$w_y = xz + 3y^2 = -1 + 3 = 2$

$w_z = xy + 4z^3 = -1 + 4 = 3$

$w_x x_t + w_y y_t + w_z z_t \Big|_{t=1,\,s=1} = (-1)(-3) + (2)(2) + (3)3$

$= 3 + 4 + 9 = \boxed{16}$

A8 $\vec{F}(x,y,z) = (3x^2, 2z, -x)$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \overset{3x^2}{P} & \overset{2z}{Q} & \overset{-x}{R} \end{pmatrix}$$

$$\cancel{i(-x_y - 2z_z}$$

$$\text{Curl}\vec{F} = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$$

$$= (0-2,\ 0-(-1),\ 0-0) = \boxed{(-2, 1, 0)}$$

A9 $(\sqrt{2}, 0, 1)_{xyz} \sim (\sqrt{2}, 0, 1)_{\text{cylindrical}} = (r, \theta, z)$

$$r = \sqrt{x^2+y^2} = \sqrt{2}$$

A10 $(-1,0,1)_{xyz} \sim (\sqrt{2}, \pi/4, \pi)_{sph} = \cancel{(\rho, \phi, \theta)}$ (ρ, θ, ϕ)

$\cancel{(\rho, \theta, \phi)}$

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+0+1} = \boxed{\sqrt{2}}$$

ϕ: $z = \rho \cdot \cos\phi \Rightarrow 1 = \sqrt{2} \times \cos\phi$

$$\Rightarrow \cos\phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \boxed{\pi/4} \text{ or }$$

θ: $x = \rho \cdot \sin\phi \cos\theta$

$$-1 = \sqrt{2} \times \frac{1}{\sqrt{2}} \cdot \cos\theta \Rightarrow \cos\theta = -1$$

$$\boxed{\theta = \pi}$$

A11

$f_x = 2xy - 2x \quad \Rightarrow \begin{cases} f_{xx} = 2y - 2 \\ f_{yy} = -4 \\ f_{xy} = 2x \end{cases}$

$f_y = x^2 - 4y$

	(0, 0)	(2, 1)
f_{xx}	-2	0
f_{yy}	-4	-4
f_{xy}	0	4
D	8	-16

saddle point

$D < 0$

$D > 0 \quad f_{xx} < 0 \Rightarrow (0, 0)$ max

$D = f_{xx} \cdot f_{yy} - f_{xy}^2$

A12

$$6\int_0^1\int_0^1 xy\left(z\Big|_0^{6-x-y}\right)dx\,dy$$

$$6\int_0^1\int_0^1 xy(6-x-y)\,dx\,dy$$

$$=6\int_0^1\int_0^1 6xy-x^2y-xy^2\,dx\,dy$$

$$=6\int_0^1\left(3x^2y-\frac{x^3y}{3}-\frac{x^2y^2}{2}\Big|_0^1\right)dy$$

$$=6\int_0^1 3y-y/3-y^2/2\,dy$$

$$=6\left(\frac{3y^2}{2}-\frac{y^2}{6}-\frac{y^3}{6}\Big|_0^1\right)$$

$$=6\left(3/2-1/6-1/6\right)=6\left(3/2-1/3\right)$$

$$=6\left(\frac{9-2}{6}\right)=7$$

??

mfm

B1. Use Lagrange Multipliers to find the maximum and minimum values of the function $f(x,y) = x^3 - 8y^3$ on the curve $x^2 + 4y^2 = 16$. [10 marks]

$$f(x,y) = x^3 - 8y^3 \qquad g(x,y) = x^2 + 4y^2 - 16 = 0$$

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y) = x^3 - 8y^3 - \lambda(x^2 + 4y^2 - 16)$$

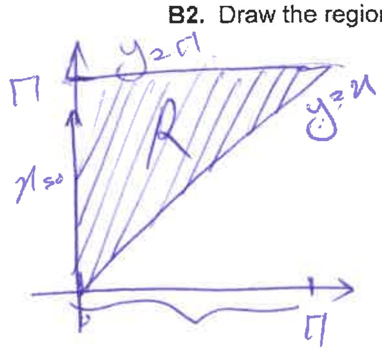
$$L_x = 3x^2 - 2\lambda x = 0 \xrightarrow{x \neq 0} 3x = 2\lambda \Rightarrow \boxed{x = \frac{2\lambda}{3}}$$

$$L_y = -24y^2 - 8\lambda y = 0 \xrightarrow{y \neq 0} -3y = \lambda \Rightarrow \boxed{y = \frac{\lambda}{-3}}$$

$$L_\lambda = -(x^2 + 4y^2 - 16) = 0 \qquad \left(\frac{2\lambda}{3}\right)^2 + 4\left(\frac{\lambda}{3}\right)^2 = 16 \Rightarrow \frac{4\lambda^2 + 4\lambda^2}{9} = 16$$

$$\Rightarrow 8\lambda^2 = 16 \times 9 \Rightarrow \lambda^2 = 18 \Rightarrow \lambda = \pm 3\sqrt{2} \Rightarrow \begin{cases} \left(\frac{2 \times 3\sqrt{2}}{3}, \frac{-3\sqrt{2}}{3}\right) \\ \left(-\frac{2 \times 3\sqrt{2}}{3}, \frac{+3\sqrt{2}}{3}\right) \end{cases}$$

B2. Draw the region of integration and evaluate $\int_0^\pi \int_x^\pi \frac{\sin y}{y} dydx$. [8 marks]



$$0 \leq x \leq \pi \qquad 0 \leq y \leq \pi$$
$$x \leq y \leq \pi \quad \Longleftrightarrow \quad 0 \leq x \leq y$$

$$\int_0^\pi \int_0^y \frac{\sin y}{y} \, dx \, dy = \int_0^\pi \frac{\sin y}{y} \left(x \Big|_0^y \right) dy$$

$$\int_0^\pi \frac{\sin y}{y} \times y \, dy = \int_0^\pi \sin y \, dy = -\cos y \Big|_0^\pi$$
$$= 2$$

B) $(2\sqrt{2}, -\sqrt{2})$

$(-2\sqrt{2}, +\sqrt{2})$

let $x=0 \Rightarrow 4y^2 = 16 \Rightarrow y = \pm 2 \Rightarrow \begin{cases} (0,2) \\ (0,-2) \end{cases}$

let $y=0 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4 \Rightarrow \begin{cases} (-4,0) \\ (4,0) \end{cases}$

$f(x,y) = x^3 - 8y^3$

$f(2\sqrt{2}, -\sqrt{2}) = 16\sqrt{2} + 16\sqrt{2} = 32\sqrt{2}$

$f(-2\sqrt{2}, +\sqrt{2}) = -16\sqrt{2} - 16\sqrt{2} = -32\sqrt{2}$

$f(0,2) = -64$

$f(0,-2) = 64$

$f(4,0) = 64$

$f(-4,0) = -64$

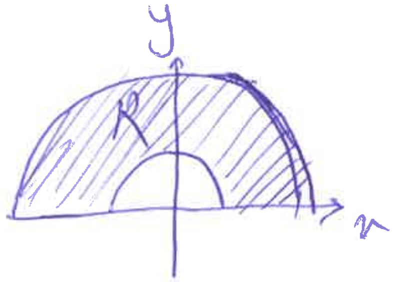
min points/values

max points/values

mgm

B3. Evaluate the double integral $\iint_R (x+2y)\,dA$,

where R is the region between $x^2+y^2=1$ and $x^2+y^2=9$ above the x-axis. [9 marks]



$1 \leq r \leq 3$ $\qquad dA \sim r\,dr\,d\theta$ $\qquad = \frac{104}{3}$

$0 \leq \theta \leq \pi$

$$\int_0^{\pi}\int_1^3 r\cos\theta + 2r\sin\theta \; r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^3 r^2\cos\theta + 2r^2\sin\theta\,dr\,d\theta = \int_0^{\pi}\left(\frac{r^3\cos\theta}{3} + \frac{2r^3}{3}\sin\theta\Big|_1^3\right)d\theta$$

$$\int_0^{\pi}\frac{26\cos\theta}{3} + \frac{52\sin\theta}{3}\,d\theta = \frac{26}{3}\sin\theta - \frac{52}{3}\cos\theta\Big|_0^{\pi}$$

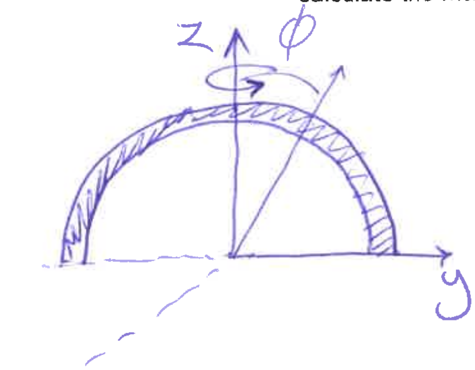
B4. Let R be the hemi-spherical shell which is between the two spheres

$x^2+y^2+z^2 = 36$ and $x^2+y^2+z^2 = 49$

and where $z \geq 0$. If the density at (x, y, z) in R is given by

$\delta(x,y,z) = 4(x^2+y^2+z^2)^{-1/2}$,

calculate the mass of R. [9 marks]



$$M = \iiint_R 4(x^2+y^2+z^2)^{-1/2}\,dV$$

Jacobian

$$M = 4\int_0^{2\pi}\int_0^{\pi/2}\int_6^7 \frac{1}{\rho}\cdot\underbrace{\rho^2\sin\phi}_{\text{Jacobian}}\,d\rho\,d\phi\,d\theta$$

$6 \leq \rho \leq 7$

$0 \leq \phi \leq \pi/2$

$0 \leq \theta \leq 2\pi$

$$M = 4\int_0^{2\pi}\int_0^{\pi/2}\int_6^7 \rho\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^{\pi/2}\sin\phi\left(\frac{\rho^2}{2}\Big|_6^7\right)d\phi\,d\theta = \frac{13}{2}\times 4\int_0^{2\pi}\left(-\cos\phi\Big|_0^{\pi/2}\right)d\theta$$

B4

$$13 \times 2 \int_0^{2\pi} (+1)\, d\theta = +2\pi \times 13 \times 2 = 52\pi$$

B6

$$\int_0^{2\pi} 4\, d\theta + 4 \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= 8\pi + 4 \int_0^{2\pi} \frac{1 + \cos 2\theta}{2}\, d\theta = 8\pi + 4\pi + \sin 2\theta \Big|_0^{2\pi}$$

$$= 12\pi$$

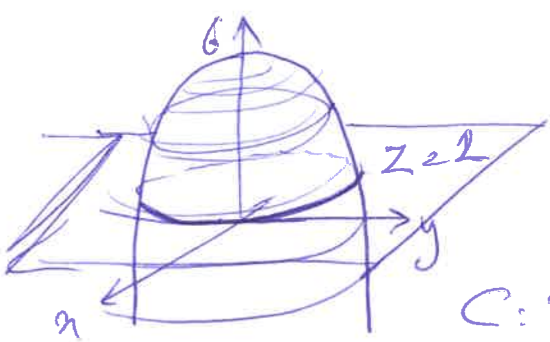
m fm

B5. Consider the vector field $F(x,y,z) = (y - x,\ z - y,\ y - x) = (y - x)\underline{i} + (z - y)\underline{j} + (y - x)\underline{k}$. Let E be the solid cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ and let S denote the surface which is the boundary of E, oriented outwards. Use Gauss' Divergence Theorem to compute the surface integral of the vector field F over S : $\iint_S F.dS = \iint_S F.\underline{n}\,dS$. [6 marks]

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_E (P_x + Q_y + R_z)\,dV$$

$$\int_0^1\int_0^1\int_0^1 -1-1\,dz\,dy\,dx = -2\iiint_E dV = -2$$

B6. Use Stokes' Theorem to evaluate $\iint_S (\nabla \times F).\underline{n}\,dS$, where $F = -y\underline{i} + xz\underline{j} + x^2yz^3\underline{k}$, S is the part of $z = 6 - x^2 - y^2$ which is above the plane $z = 2$ and $\underline{n}$ is the outer normal vector to S. [10 marks]



$$\iint_S (\nabla\times F)\cdot dS = \int_C F\cdot dr$$

$$C: r(\theta) = (2\cos\theta, 2\sin\theta, 2)$$

$$z = 2 \Rightarrow 2 = 6 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$$

$$\int_C F\cdot dr = \int_0^{2\pi} F(r(\theta))\cdot r'(\theta)\,d\theta$$

$$= \int_0^{2\pi} (-2\sin\theta, 4\cos\theta, 32\cos^3\theta\sin\theta)\cdot(-2\sin\theta, 2\cos\theta, 0)\,d\theta$$

$$= \int_0^{2\pi} 4\sin^2\theta + 8\cos^2\theta\,d\theta = \int_0^{2\pi} 4 + 4\cos^2\theta\,d\theta$$

mjm

This page to be used for continuation of your answers, if you need it. Be sure to indicate on any given question if your solution extends here.