MATH232 D100, Fall 2019

7.1 Basis and Dimension

(based on notes from Dr. J. Manuch)

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Recall:

- If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_s}$ are vectors in R^n, then $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_s}\}$ is the set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_s}$:

$$\text{span}\{\mathbf{v_1}, \ldots, \mathbf{v_s}\} = \{c_1\mathbf{v_1} + \cdots + c_s\mathbf{v_s} : \; c_1, c_2, \ldots, c_s \in R\}$$

- $\text{span}\{\mathbf{v_1}, \ldots, \mathbf{v_s}\}$ is a subspace of R^n

Example. Consider a subspace $W = \text{span}\{(1,0,0), (0,1,0), (1,1,0)\}$ of R^3. Show that $W = \text{span}\{(1,0,0), (0,1,0)\} = V$

$V \subseteq W$: Let $\underline{v} \in V$. Then $\underline{v} = t_1(1,0,0) + t_2(0,1,0)$, for some t_1, t_2.

so $\underline{v} = t_1(1,0,0) + t_2(0,1,0) + 0\cdot(1,1,0)$.

Thus $\underline{v} \in \underline{W}$ and $V \subseteq W$.

$W \subseteq V$: Let $\underline{w} \in W$. Then $\underline{w} = t_1(1,0,0) + t_2(0,1,0) + t_3(1,1,0)$

$\underline{w} = t_1(1,0,0) + t_2(0,1,0) + t_3\big(1\cdot(1,0,0) + 1\cdot(0,1,0)\big)$

$= (t_1+t_3)(1,0,0) + (t_2+t_3)(0,1,0)$

$\in \text{span}\{(1,0,0), (0,1,0)\}$. Thus $W \subseteq V$.

Definition (basis)

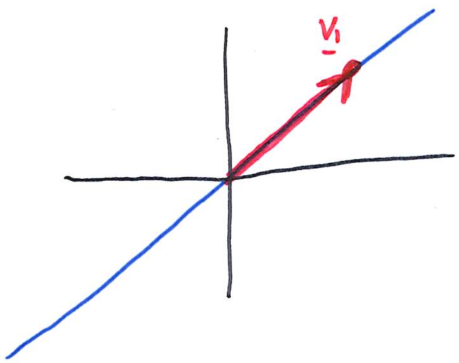
A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ in a subspace V of R^n is called a **basis** for V if

(i) $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\} = V$; and

(ii) The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ is *linearly independent*.

Equivalently, the span of any proper subset of $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_s\}$ is not equal to V

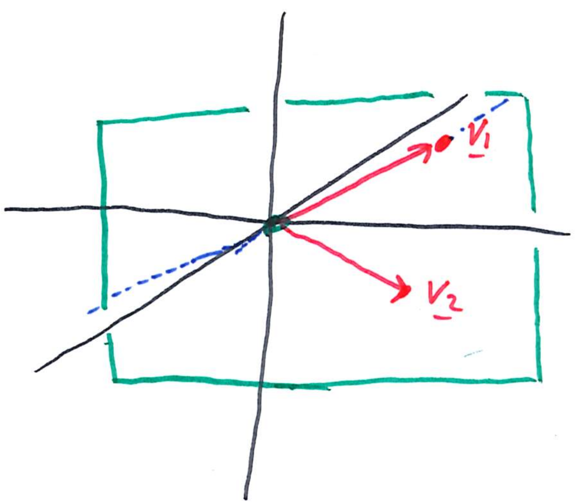
Example. Find a basis for subspace L, where L is a line through the origin in R^n.



Just select any non-zero vector $\underline{v}_1$ from L.

Now $\{\underline{v}_1\}$ is a basis of L since $\text{span}\{\underline{v}_1\} = L$ and $\{\underline{v}_1\}$ is lin. indep.

Example. Find a basis for subspace P, where P is a plane through the origin in R^n.



1) select any $\underline{v}_1 \in P$, $\underline{v}_1 \neq \underline{0}$

2) select any $\underline{v}_2 \in P$, $\underline{v}_2 \notin \text{span}\{\underline{v}_1\}$

Example. Show that the set $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ is a basis of R^n.

1) $\text{Span}\{\underline{e}_1, \ldots, \underline{e}_n\} = \mathbb{R}^n$

2) $\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n\}$ are lin. indep.

$$\underbrace{\begin{bmatrix} | & | & & | \\ \underline{e}_1 & \underline{e}_2 & \cdots & \underline{e}_n \\ | & | & & | \end{bmatrix}}_{= I_n} \underline{x} = \underline{0}$$ has no nontrivial solutions

$= I_n$ is already row-reduced and every column has a leading 1.

Note: The set $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ is called the **standard basis** for R^n. The subspace $\{\mathbf{0}\}$ has no basis.

A condition for linear independence

Theorem (7.1.2)

If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_s}\}$ is a set of two or more non-zero vectors in R^n, then S is linearly dependent if and only if for some i, vector $\mathbf{v_i}$ is a linear combination of its predecessors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{i-1}}$.

Equivalently, the S is linearly independent if and only if for every $i = 2, \ldots, s$, $\mathbf{v_i}$ cannot be expressed as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{i-1}}$.

Proof.

($\Leftarrow$) Suppose $\underline{v_i} = t_1 \underline{v_1} + t_2 \underline{v_2} + \cdots + t_{i-1} \underline{v_{i-1}}$

Then

$$t_1 \underline{v_1} + t_2 \underline{v_2} + \cdots + t_{i-1} \underline{v_{i-1}} + (-1) \underline{v_i} + 0 \underline{v_{i+1}} + 0 \underline{v_{i+2}} + \cdots + 0 \underline{v_n} = \underline{0}$$

and not all coefficients are zero! Thus they are lin. dependent.

($\Rightarrow$) Suppose $t_1 \underline{v_1} + t_2 \underline{v_2} + \cdots + t_n \underline{v_n} = \underline{0}$, not all t_i's are zero.

Let i be the largest index such that $t_i \neq 0$.

we solve for $\underline{v_i}$

$$\underline{v_i} = \left(-\frac{t_1}{t_i}\right)\underline{v_1} + \left(-\frac{t_2}{t_i}\right)\underline{v_2} + \cdots + \left(-\frac{t_n}{t_i}\right)\underline{v_n}$$

$\square$

Example. Show that vectors $\mathbf{v_1} = (3, 0, 0)$, $\mathbf{v_2} = (0, 4, 6)$ and $\mathbf{v_3} = (0, -2, 2)$ are linearly independent.

1) $\{\underline{v}_1\}$ is lin. indep. since $\underline{v}_1 \neq \underline{0}$

2) $\{\underline{v}_1, \underline{v}_2\}$ is lin. indep since $\underline{v}_2$ is not a scalar multiple of $\underline{v}_1$

3) $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is lin. indep. since $\{\underline{v}_1, \underline{v}_2\}$ is lin. indep. and $\underline{v}_3$ is not a linear comb. of $\underline{v}_1, \underline{v}_2$

Suppose $\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} = t_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$. Looking at the first coordinate, we must have $t_1 = 0$.

Also $t_2 = 0$ since $\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$ is not a scalar multiple of $\begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$

The existence of a basis

Theorem (7.1.3)

If $V \neq \{\mathbf{0}\}$ is a (non-zero) subspace of R^n, then there exists a basis for V that has at most n vectors.

Proof.

We will give a procedure for constructing a set in V of linearly independent vectors until it spans the set V:

1) Let $\underline{v}_1$ be any non-zero vector in V.
   Set $i=1$

2) Given linearly independent vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_i$ in V
   If span $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_i\} = V$, then $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_i\}$ is a basis of V. STOP.
   Else let $\underline{v}_{i+1}$ be any vector in $V -$ span $\{\underline{v}_1, \dots, \underline{v}_i\}$

3) Set $i = i+1$ and go to 2)

Procedure must stop with $i \leq n$ by Obs 3,4 of text.

Theorem (7.1.4)

Suppose V is a non-zero subspace of R^n. Then every basis for V contains the same number of vectors.

Hence,

- Every basis for a line through origin of R^n has one vector.
- Every basis for a plane through origin of R^n has two vectors.
- Every basis for a hyperplane in R^n has $n-1$ vectors.
- Every basis for R^n has n vectors.

Definition (dimension)

The **dimension** of a non-zero subspace V of R^n, denoted $\dim(V)$, is the number of vectors in a basis for V. We also say that the zero subspace $\{\mathbf{0}\}$ has dimension 0.

Example. Show that the non-zero rows of a matrix in row echelon form are linearly independent.

$$A = \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 rows form a basis of row(A)

No row is a linear comb. of rows below it.

Canonical bases from row reduction

When there are many ways to describe an object, we often make a rule that generates one “standard” or **canonical** description. For example, in the military, the canonical format for time of day is **hh:mm:ss**.

There are many different bases which span the same subspace V. With Gauss-Jordan elimination, we can get canonical bases for two subspaces associated with a matrix A.

Recall that applying elementary row operations does not affect either

1. the **row space**, $\mathrm{row}(A)$, the set of linear combinations of the rows of A, or
2. the **null space**, $\mathrm{null}(A)$, the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Let B be the (unique!) *reduced row echelon form* of A.

1. The non-zero rows of B can serve as the **canonical basis for** $\mathrm{row}(A)$.
2. Solving $A\mathbf{x} = \mathbf{0}$, using B in the usual way results in the **canonical basis for** $\mathrm{null}(A)$.

Canonical bases from row reduction

Example. See Example 7 in Section 2.2 (P. 56)

Find canonical bases for the row space and null space of $A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$.

Solution: The RREF of A is $B = \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

1. The row space of A has dimension 3, and the canonical basis for row(A) is

$$\{\, (1,3,0,4,2,0),\ (0,0,1,2,0,0), (0,0,0,0,0,1)\,\}.$$

2. Solving $A\mathbf{x} = \mathbf{0}$ as usual, we parameterize the free variables $x_2 = r$, $x_4 = s$, $x_5 = t$ to get the general solution

$$\begin{aligned} \mathbf{x} &= (-3r - 4s - 2t, r, -2s, s, t, 0), \qquad (-\infty < r, s, t < \infty) \\ &= r(-3,1,0,0,0,0) + s(-4,0,-2,1,0,0) + t(-2,0,0,0,1,0). \end{aligned}$$

The null space of A has dimension 3, and the canonical basis for null(A) is

$$\{\, (-3,1,0,0,0,0),\ (-4,0,-2,1,0,0),\ (-2,0,0,0,1,0)\,\}$$

Dimension of the row space and null space of A

Example. See Example 7 in Section 2.2 (P. 56)

$$\text{The RREF of } A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \text{ is } B = \begin{array}{c} \begin{array}{cccccc} \downarrow & & \downarrow & & & \downarrow \end{array} \quad \text{(leading)} \\ \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} & \uparrow & & \uparrow & \uparrow & \end{array} \quad \text{(free)} \end{array}.$$

The RREF form of A has 3 leading 1s and 3 "free" columns.

Theorem

Suppose we reduce an $n \times m$ matrix A to its RREF B. Suppose B has s leading 1s. Then

1. $\dim(\text{row}(A)) = s$ *(the number of leading columns in B), and*
2. $\dim(\text{null}(A)) = m - s$ *(the number of free variables in $[B|0]$).*