CONCORDIA UNIVERSITY **Department of Mathematics and Statistics**

Course	Number	Section(s)
Mathematics	**MATH 364**	**AA**

Examination	Date	Time	Pages
Final	**April 2011**	**3 hours**	**4**

Instructors	Course Examiner
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Special Instructions: Calculators permitted. Lined paper booklets.

READ THE QUESTIONS CAREFULLY !!! SHOW ALL WORK !!!
JUSTIFY ALL STEPS !!! GOOD LUCK !!!

MARKS: marks for each problem are shown in front of the problems. Maximum total is 100.

↓MARKS

5 **Problem 1 :** Prove that the sentence below is always true:

$$[(\alpha \implies \beta) \wedge (\beta \implies \alpha)] \implies (\alpha \vee \beta) .$$

Solution: Unfortunately it is not: when $\alpha = 0$ and $\beta = 0$ we obtain $1 \implies 0$ which is false.

15 **Problem 2 :** Let $f : \mathbb{R} \to \mathbb{R}$. Use quantifiers to write:

(a) The definition of uniform continuity of f on $\mathbb{R}$;
(b) The negation of the above definition;
(c) Is the function $f(x) = \sin(x)$ uniformly continuous on $\mathbb{R}$?
(d) Is the function $f(x) = \frac{1}{x^8}$ uniformly continuous on $(0, +\infty)$?

Solution: (a)

$$\forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{x,y\in\mathbb{R}} \ |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon .$$

(b)

$$\exists_{\varepsilon>0} \ \forall_{\delta>0} \ \exists_{x,y\in\mathbb{R}} \ |x-y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon .$$

(c) Using Mean Value Th.:

$$|\sin(x) - \sin(y)| = |\cos(c)||(x-y| \leq 1 \cdot |x-y| ,$$

so function sin satisfies Lipschitz condition on $\mathbb{R}$ and thus is uniformly continuous on $\mathbb{R}$.

(d) Using the Theorem: If function f is uniformly continuous and the sequence (x_n) is Cauchy, then the sequence $(f(x_n))$ is also Cauchy : Assume that $f(x) = 1/x^8$ is uniformly continuous on $(0, +\infty)$. Consider sequence $x_n = 1/n$, $n = 1, 2, 3, \ldots$ which belongs to the domain and is Cauchy (since it converges to 0). The sequence $f(x_n) = n^8$ is not Cauchy (since it diverges to $+\infty$). In view of the Theorem, this contradicts our assumption. Thus, f is not uniformly continuous on $(0, +\infty)$.

15 **Problem 3 :** Let a set $A \subset \mathbb{R}$ be bounded and let $\alpha = \sup A$.

(a) Prove that

$$\forall_{\varepsilon>0} \exists_{a\in A} \ \alpha - \varepsilon < a \leq \alpha \ .$$

(b) Construct a sequence of elements $a_n \in A$ such that $\lim_{n\to\infty} a_n = \alpha$.

(c) Let

$$A = \left\{ \left(1 + \frac{1}{n}\right)^{\frac{1}{n}} : n = 1, 2, \dots \right\} \ .$$

Find $\inf A$.

Solution: (a) Standard: can be found in the book.

(b) Using part (a): for any $n \geq 1$ we can find an $a_n \in A$ such that $\alpha - 1/n < a_n \leq \alpha$. Then, $a_n \to \alpha$ by Squeeze Th.

(c) We will prove that $\inf A = 1$. Obviously, $\sqrt[n]{1 + 1/n} \geq 1$ for all $n \geq 1$ so 1 is a lower bound for A. We will show that $\sqrt[n]{1 + 1/n} \to 1$ as $n \to \infty$. In view of (a), this will show that $\inf A = 1$. We have

$$1 \leq \sqrt[n]{1 + 1/n} \leq 1 + 1/n \ ,$$

so the convergence follows by Squeeze Th.

15 **Problem 4 :**

(a) State the Cauchy definition of a limit of $f : \mathbb{R} \to \mathbb{R}$ at a point x_0;

(b) Prove, using Cauchy definition, that $\lim_{x\to 5} x^2 + x + 5 = 35$.

(c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(1) = 2013$, then there exists a $\delta > 0$ such that $f(x) \leq 2014$ for x in the δ neighbourhood of 1.

Solution: (a) If L is the value of the limit:

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x\in\mathbb{R}} \ 0 < |x - x_0| < \delta \Longrightarrow |f(x) - L| < \varepsilon \ .$$

(b) We want to show:

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x\in\mathbb{R}} \ 0 < |x - 5| < \delta \Longrightarrow |x^2 + x + 5 - 35| < \varepsilon \ .$$

We want to write $x^2 + x - 30 = (x - 5)(ax + b)$. It is easy to see that $a = 1$ and $b = 6$. Let us fix $\varepsilon > 0$. We want to find $\delta > 0$ such that $|x - 5| < \delta$ implies $|x - 5||x + 6| < \varepsilon$. To estimate $|x + 6|$ we assume that $\delta \leq 1$. Then, $|x - 5| < \delta$ implies $4 < x < 6$ and $|x + 6| < 12$. Let us set $\delta = \min\{1, \varepsilon/12\}$. Then,

$$|x - 5||x + 6| < |x - 5|12 < \delta \cdot 12 \leq \varepsilon \ .$$

For arbitrary $\varepsilon > 0$ we found $\delta > 0$ which satisfies the conditions of the definition. We proved that the limit is 35.

(c) Let us rewrite the definition of continuity in this particular case:

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x\in\mathbb{R}} \ |x - 1| < \delta \Longrightarrow |f(x) - 2013| < \varepsilon \ .$$

The last inequality says $2013 - \varepsilon < f(x) < 2013 + \varepsilon$. Let $\varepsilon = 1$. We can find $\delta > 0$ such that

$$|x - 1| < \delta \Longrightarrow 2013 - 1 < f(x) < 2013 + 1 \ .$$

This means that in the interval $(1 - \delta, 1 + \delta)$ function f satisfies $f(x) < 2014$.

↓MARKS

5 **Problem 5 :** Let $x_n = (-1)^n + 2^{(-1)^n}$, $n = 1, 2, 3, \ldots$ Find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

Solution: We have two convergent subsequences: for odd n we have

$$x_n = -1 + 1/2 = -1/2 \to -1/2 \ ,$$

and for for even n we have

$$x_n = 1 + 2 = 3 \to 3 \ .$$

Every other convergent subsequence must be a subsequence of one of them starting from some index. Thus, $\limsup_{n\to\infty} x_n = 3$ and $\liminf_{n\to\infty} x_n = -1/2$.

10 **Problem 6 :** Prove that if a sequence $\{x_n\}$ is divergent, then we have strict inequality

$$\liminf_{n\to\infty} x_n < \limsup_{n\to\infty} x_n \ .$$

Solution: I will assume that $\{x_n\}$ is bounded. The general case is done similarly. By Bolzano-Weierstrass Th. $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$. Let $x_{n_k} \to a$. But $\{x_n\}$ is not convergent so in particular it does not converge to a. We write the negation of the definition of convergence to a:

$$\exists_{\ \varepsilon>0} \ \forall_{\ N\geq 1} \ \exists_{\ n\geq N} \ |x_n - a| \geq \varepsilon \ .$$

This means that an infinite number of elements of $\{x_n\}$ is outside interval $(a - \varepsilon, a + \varepsilon)$. We can apply Bolzano-Weierstrass Th. to these elements. Thus, we obtain a convergent subsequence of $\{x_n\}$ convergent to a limit $b \neq a$. Say $a < b$. Then,

$$\liminf_{n\to\infty} x_n \leq a < b \leq \limsup_{n\to\infty} x_n \ .$$

15 **Problem 7 :** (a) Is the function below differentiable at $x_0 = 0$?

$$f(x) = \begin{cases} x^2 \cos\left(\frac{2013}{x}\right) & \text{if} \quad x \neq 0 \ ; \\ 0 & \text{if} \quad x = 0 \ . \end{cases}$$

(b) If it is, is its derivative f' continuous at $x_0 = 0$?

Solution: (a) We have

$$f'(0) = \lim_{h\to 0} \frac{h^2 \cos\left(\frac{2013}{h}\right)}{h} = \lim_{h\to 0} h \cos\left(\frac{2013}{h}\right) = 0 \ .$$

(b) Outside 0 we have:

$$f'(x) = 2x \cos\left(\frac{2013}{x}\right) + x^2 \left(-\sin\left(\frac{2013}{x}\right)\right) \frac{-2013}{x^2} = 2x \cos\left(\frac{2013}{x}\right) + 2013 \sin\left(\frac{2013}{x}\right) \ .$$

For $x \to 0$ the first term converges to 0 but the second term does not have a limit (every number in $[-2013, 2013]$ is a partial limit). f' is not continuous at 0.

10 **Problem 8 :** (a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that $f' \leq 0$ if and only if f is decreasing.

Solution: Let $f' \leq 0$. Let $x < y$. Then, by Mean Value Th.,

$$f(y) - f(x) = f'(c)(y - x) \leq 0 \ ,$$

and f is decreasing.

Let f be decreasing. We have

$$f'(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \leq 0 \ ,$$

since $f(x+h) - f(x) \leq 0$ for $h > 0$.

10 **Problem 9 :** Prove that for $x, y \geq 1$ we have

$$|\arctan x - \arctan y| \leq \frac{1}{2}|x - y| \ .$$

Solution: By Mean Value Th.

$$|\arctan x - \arctan y| = \left| \frac{1}{1 + c^2} \right| |x - y| \leq \frac{1}{2}|x - y| \ ,$$

since c is between x and y so $1 \geq c$.

10 **Bonus question:** How many zeros has $f(x) = 4x^3 - 32x^2 + 79x - 60$ in the interval $[0, 5]$?

Solution: First, let us notice that $f(0) = -60 < 0$ and $f(5) = 35 > 0$ so there is at least one zero inside. To refine our search we have two methods:

(1) just check the values of f at other points, for example $f(2) = 2 > 0$ and $f(3) = -3 < 0$. This gives us sequence of signs $-, +, -, +$ proving (by Intermediate Value Th.) existence of at least three zeros in $[0, 5]$. Since polynomial of order three cannot have more zeros, this is the answer.

(2) a little more organized search: the derivative of f is

$$f'(x) = 12x^2 - 64x + 79 \ ,$$

with zeros $x_1 \sim 1.94$ and $x_2 \sim 3.39$. Now, we can check that $f(x_1) \sim 2.03$ and $f(x_2) = -4.1$ which gives the same sequence of signs and the same answer.