MAT 1320 Review sheet for Midterm 1

- Functions: domain, range, composite function $(f \circ g)(x) = f(g(x))$, inverse function, basic functions from high school (exponential, log, trigonometric ...)

- Limits:
$$\begin{cases} \lim_{x \to a} f(x) = L, & L \text{ is a number} \\ \lim_{x \to \infty} f(x) = L, & L \text{ is a number} \\ \lim_{x \to a} f(x) = \infty, & \text{infinite limit (limit doesn't exist)} \end{cases}$$

- calculation regarding limits:
  - limit laws: $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

$$\lim_{x \to a} [f(x) g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0.$$

  - Direct substitution rule
  - $\lim_{x \to \infty} \frac{f(x)}{g(x)}$, when $f(x)$ and $g(x)$ are polynomials, the general rule is that dividing both numerator and denominator by the term with the highest power.

- Derivative:
  - Definition: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
  - Rules regarding calculating derivative: summation rule, difference rule, product rule ...

③ $\begin{cases} \text{chain rule} \\ \text{implicit differentiation} \\ \text{related rates.} \end{cases}$

Sample questions:

- Find the domain of each function

(a) $\dfrac{1-e^{x^2}}{1-e^{1-x^2}}$ (b) $\dfrac{1+x}{e^{\cos x}}$

Solution: (a): $1-e^{1-x^2} \neq 0 \Rightarrow 1 \neq e^{1-x^2} \Rightarrow 1-x^2 \neq 0 \Rightarrow x \neq \pm 1$

(b): $e^{\cos x} \neq 0$ is always satisfied. Therefore the domain is $x \in R$.

- Evaluate the limit:

- $\lim\limits_{t\to 0} \left(\dfrac{1}{t\sqrt{1+t}} - \dfrac{1}{t} \right)$

Solution: $\lim\limits_{t\to 0} \left(\dfrac{1}{t\sqrt{1+t}} - \dfrac{1}{t} \right) = \lim\limits_{t\to 0} \dfrac{1-\sqrt{1+t}}{t\sqrt{1+t}} = \lim\limits_{t\to 0} \dfrac{(1-\sqrt{1+t})(1+\sqrt{1+t})}{t\sqrt{1+t}(1+\sqrt{1+t})}$

$= \lim\limits_{t\to 0} \dfrac{1-(1+t)}{t\sqrt{1+t}(1+\sqrt{1+t})} = \lim\limits_{t\to 0} \dfrac{-t}{t\sqrt{1+t}(1+\sqrt{1+t})} = \lim\limits_{t\to 0} \dfrac{-1}{\sqrt{1+t}(1+\sqrt{1+t})} = \dfrac{-1}{\sqrt{1}\cdot 2}$

$= -\dfrac{1}{2}$

- $\lim\limits_{t\to\infty} \dfrac{\sqrt{t}+t^2}{2t-t^2} = \lim\limits_{t\to\infty} \dfrac{\frac{\sqrt{t}}{t^2}+1}{\frac{2}{t}-1} = \lim\limits_{t\to\infty} \dfrac{\frac{1}{t^{\frac{3}{2}}}+1}{\frac{2}{t}-1} = -1$

solution

- Differentiate $y = e^p(p+p\sqrt{p})$

Solution: $y' = (e^p)'(p+p\sqrt{p}) + e^p(p+p\sqrt{p})'$

$= e^p(p+p\sqrt{p}) + e^p(1+\frac{3}{2}p^{\frac{1}{2}})$

$= e^p(p+\frac{3}{2}p^{\frac{1}{2}} + p^{\frac{3}{2}} + 1)$

③ • Differentiate $y = e^{-x^2+\sqrt{x}}$

$$y' = e^{-x^2+\sqrt{x}} \cdot \left(-x^2+\sqrt{x}\right)' = e^{-x^2+\sqrt{x}} \cdot \left(-2x + \frac{1}{2}x^{-\frac{1}{2}}\right)$$

• Find $\frac{dy}{dx}$ by implicit differentiation:

$$x^2 - 4xy + y^2 = 4$$

Solution: Note that $\frac{dy}{dx}$ means y'.

$$\left(x^2 - 4xy + y^2\right)' = 4'$$

$$2x - 4(xy)' + (y^2)' = 0$$

Here y is a function of x. Always remember that.

Therefore $2x - 4\left[x'y + xy'\right] + 2y \cdot y' = 0$

$$\Rightarrow \quad 2x - 4(y + xy') + 2y \cdot y' = 0$$

$$\Rightarrow \quad 2x - 4y - 4xy' + 2y \cdot y' = 0 \Rightarrow 2(x-2y) = 2y'(2x-y)$$

$$\Rightarrow y' = \frac{x-2y}{2x-y}$$

• Use logarithmic differentiation to find the derivative. $y = x^{\sin(x)}$

Solution: $\ln y = \ln x^{\sin(x)} \Rightarrow \ln y = \sin(x) \cdot \ln x$

$$\Rightarrow \quad (\ln y)' = (\sin(x)\ln x)'$$

$$\Rightarrow \quad \frac{y'}{y} = (\sin(x))'\ln x + \sin(x)(\ln x)'$$

$$\Rightarrow \quad \frac{y'}{y} = \cos(x)\ln x + \sin(x) \cdot \frac{1}{x}$$

$$\Rightarrow \quad y' = y\left(\cos(x)\ln x + \frac{\sin(x)}{x}\right)$$

$$\Rightarrow \quad y' = x^{\sin(x)}\left(\cos(x)\ln x + \frac{\sin(x)}{x}\right)$$

④

- Related rates questions (please see lecture notes and DGD questions. Too long to put here.)

Formula that you need to remember:

- Laws of exponents : $b^{x+y} = b^x b^y$, $b^{x-y} = \frac{b^x}{b^y}$, $(b^x)^y = b^{xy}$, $(ab)^x = a^x b^x$
- Laws of logarithms: $\log_b(xy) = \log_b x + \log_b y$, $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$, $\log_b(x^r) = r\log_b x$
- $\log_b b^x = x$, $b^{\log_b x} = x$, $e^{\ln x} = x$, $\log_b x = \frac{\ln x}{\ln b}$
- $(c)' = 0$, $(x^n)' = nx^{n-1}$, $(cf(x))' = cf'(x)$

  $(e^x)' = e^x$, $(a^x)' = a^x \ln a$

  $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$

  $(\ln x)' = \frac{1}{x}$, $(\log_b x)' = \frac{1}{x\ln b}$