

MAT 1341 – Midterm 2 Practice Exam

1. Let $S = \{(1, 1, 0, 0), (0, 0, -3, 1)\}$. Which of the following pairs of vectors could be added to S in order to extend S to a basis of \mathbb{R}^4 ?

$$W = \{(1, 0, 0, 0), (0, 1, 0, 0)\}$$

$$X = \{(1, 0, 0, 0), (0, 0, 0, 1)\}$$

$$Y = \{(0, 0, 1, 0), (0, 0, 0, 1)\}$$

$$Z = \{(0, 1, 0, 0), (0, 0, 1, 0)\}$$

- A. W and X .
- B. W and Y .
- C. W and Z .
- D. X and Y .
- E. X and Z .
- F. Y and Z .

Write $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$ and observe that the given vectors can give us a pivot in column 1 or 2, and column 3 or 4. So we need v_1 to give us another pivot in column 1 or 2, and v_2 to give us another pivot in column 3 or 4.

2. Suppose A is a 5×6 matrix with $\text{rank} A = 4$. What is $\dim \text{Null}(A)$?

- A. 1
- B. 2
- C. 3
- D. 4
- E. 5
- F. None of the above.

$$5 \begin{bmatrix} \boxed{\cdot} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \boxed{\cdot} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

2 columns with no pivots

Recall that $\text{rank} A + \dim \text{Null} A = \# \text{ of cols}$,
 so $\dim \text{Null} A = 2$

3. Let V be a vector space and let $u, v, w \in V$ be such that $\{u, v\}$ is linearly independent, but $\{u, v, w\}$ is linearly dependent. Which two of the following statements must be true?

- F** I. $u \in \text{span}\{v, w\}$
- T** II. $w \in \text{span}\{u, v\}$
- F** III. $\{u + w, v + w\}$ is linearly dependent.
- T** IV. $\text{rank}[u \ v \ w] = 2$

- A. I & II
- B. I & III.
- C. I & IV.
- D. II & III.
- E. II & IV.**
- F. III & IV.

$u \in \text{span}\{v, w\}$: could be false if $w = \underline{0}$ because $\text{span}\{v, \underline{0}\} = \text{span}\{v\}$, and we know $u \notin \text{span}\{v\}$ since $\{u, v\}$ LI.

$w \in \text{span}\{u, v\}$: true since if $w \notin \text{span}\{u, v\}$ then $\{u, v, w\}$ would be LI, and it's not.

$\{u + w, v + w\}$ LD: false if $w = \underline{0}$ because then this is just $\{u, v\}$, which is LI.

$\text{rank}[u \ v \ w] = 2$: since $\{u, v\}$ LI, we know that $\text{rank}[u \ v] = 2$. Then $\{u, v, w\}$ LD means $w \in \text{span}\{u, v\}$, so $[u \ v | w]$ is consistent, so $\text{rank}[u \ v | w] = \text{rank}[u \ v]$.

4. Let $S = \{2x + 1, x^3 + 2\}$ be a set of polynomials in \mathbb{P}_3 , the vector space of polynomials of degree at most 3.

- Set up a matrix A whose rows are the coordinate vectors of each of the polynomials in S with respect to the standard ordered basis $\{x^3, x^2, x, 1\}$ of \mathbb{P}_3 .
- Use your matrix A to extend S to a basis \mathcal{B} for \mathbb{P}_3 using an appropriate algorithm from class.
- Write $x^3 + 2x^2 + 4x - 1$ as a linear combination of the vectors in your basis \mathcal{B} .

a) $A = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$
 coeffs of x^3 x^2 x 1

Please note: sometimes we put the polynomials into columns (e.g. to test LI), but to extend to a basis for \mathbb{P}_3 we will put them in the rows.

b) $\begin{bmatrix} 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ \downarrow v_1 \\ \downarrow v_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{matrix} R_2 \\ R_1 \\ \downarrow v_1 \\ \downarrow v_2 \end{matrix}$

We can get a pivot in columns 1 and 3, so we need v_1 to give a pivot in col 2 and v_2 to give a pivot in col 4.

Let $v_1 = x^2$, $v_2 = 1$. $\mathcal{B} = \{2x+1, x^3+2, x^2, 1\}$

Note: other answers can also be correct if they use the same reasoning.

c) Now write these polynomials in the columns and solve

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 4 \\ 2 & 0 & 1 & 1 & -1 \end{array} \right] \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -3 \end{array} \right] \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 - 2R_1 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array} \right] \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 - R_3 \end{matrix}$$

I wrote the cols in a different order for convenience

$\therefore x^3 + 2x^2 + 4x - 1 =$

3

$1(x^3+2) + 2x^2 + 2(2x+1) - 5 \cdot 1$

5. Let

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -8 \\ 3 & 7 & 7 & -2 \end{bmatrix}$$

- What is the dimension of $\text{Col}(A)$?
- Find a basis for $\text{Col}(A)$.
- Find a basis for $\text{Null}(A)$.

a) Row reduce enough to find rank A :

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -8 \\ 3 & 7 & 7 & -2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & -4 \\ 0 & 4 & -4 & -8 \\ 0 & -2 & 10 & -8 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_4 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 4 & -4 & 8 \\ 0 & -2 & 10 & -8 \end{bmatrix} -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 6 & 0 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ R_3 - 4R_2 \\ R_4 + 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & 0 \end{bmatrix} \frac{1}{4}R_3 \sim \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix} \begin{matrix} R_4 - 6R_3 \end{matrix}$$

There are 4 pivots, so $\dim \text{Col} A = 4$.

b) We pick all the original columns where A has a pivot, so $\mathcal{B} = \{(1, 2, 0, 3), (1, 1, 4, 7), (-1, 0, -4, 7), (2, 0, -8, -2)\}$ is a basis for $\text{Col} A$.

c) The solution set to $A\vec{x} = \vec{0}$ gives a basis for $\text{Null} A$ (when in RREF). But here, there is only the trivial solution, so a basis for $\text{Null} A$ is \emptyset , the empty set ($\text{Null} A = \{\vec{0}\}$).

6. Indicate if each of the following statements is (always) true or is (possibly) false. Put your answers in the boxes indicated and show your work in the spaces below each question.

- If you indicate that the statement is (possibly) false, **you must give an explicit counterexample with numbers!**
- If you indicate that the statement is (always) true, you must give a clear explanation supported with results from class.

a) If $\{u, v\}$ is a linearly independent set of vectors in a vector space and $w \in \text{span}\{u, v\}$, then $[u \ v \ | \ w]$ is inconsistent.

On the contrary, if $w \in \text{span}\{u, v\}$, then there is a sol'n to $c_1 u + c_2 v = w$, so $[u \ v \ | \ w]$ is consistent.

ANSWER : FALSE

b) If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1+1 \\ 0 & 1 & 1 & 1-1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so these are inverses

ANSWER : TRUE

6 (continued).

- c) If $\{v_1, \dots, v_k\}$ is a linearly dependent set of vectors in \mathbb{R}^n and $A = [v_1 \ \dots \ v_k]$, then $Ax = b$ has a solution for any $b \in \mathbb{R}^n$.

False: let $v_1 = \underline{0}$ and $k=1$. Then $\{v_1\}$ is LD and $[v_1 | b] = [0 | b]$ is inconsistent whenever $b \neq \underline{0}$.

ANSWER:

FALSE

- d) If A is an $m \times n$ matrix and $\text{Null}(A) = \{0\}$, then $\text{rank} A = n$.

$\dim\{0\} = 0$ and recall that

$$\text{rank} A + \dim \text{Null} A = \# \text{ of cols}$$

$$\begin{array}{ccc} \parallel & & \parallel \\ 0 & & n \end{array}$$

so $\text{rank} A = n$.

ANSWER :

TRUE