Practice Questions for Midterm 2

1. Let $G = (V, E)$ be a graph with n vertices and let $\delta(G) = \min_{v\in V}\{\deg v\}$. Prove the following statements.

   (a) If $\deg u + \deg v \geq n - 1$ for every two nonadjacent vertices u and v of G, then G is connected (by contradiction).

   (b) If $\delta(G) \geq (n-1)/2$, then G is connected (by contradiction).

   (c) Prove that the bound in (a) is sharp by finding a disconnected graph on n vertices such that $\deg u + \deg v = n - 2$ for each even positive integer n.

2. Let T be a tree.

   (a) Suppose $\deg v \in \{1, 5\}$ for each vertex v of T, and T has 25 vertices of degree 5. How many vertices does T have?

   (b) Suppose T has 21 vertices and $\deg v \in \{1, 3, 5, 6\}$ for each vertex v of T. If T has exactly 15 leaves and one vertex of degree 6, how many vertices of T have degree 5?

3. A tree T with 35 vertices has 25 leaves, two vertices of degree 2, three vertices of degree 4, two vertices of degree 6 and three vertices of degree x. What is x?

4. Prove, by induction on n, that if a connected graph has n vertices and $n-1$ edges, then it is a tree.

5. Prove or disprove:

   (a) If a planar graph contains a triangle, then its chromatic number is 3.

   (b) If there is a 4-colouring of a graph G, then $\chi(G) = 4$.

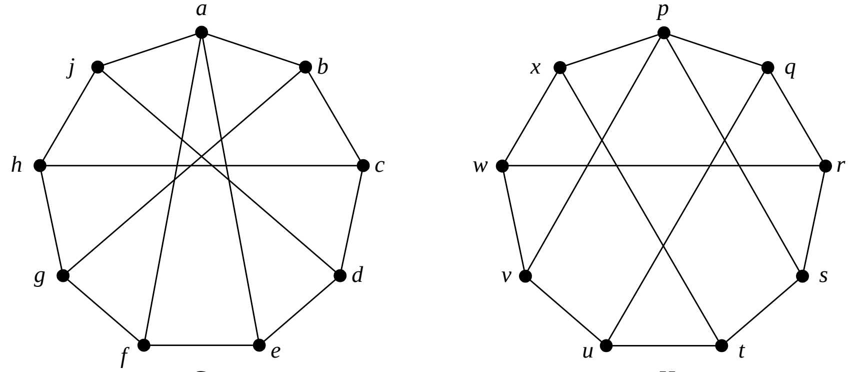
   (c) If we can prove that a graph G has no 3-colouring, then $\chi(G) = 4$.

   (d) If $\chi(G) \leq 4$, then G is planar.

   (e) If G contains a subgraph isomorphic to K_r, then $\chi(G) \geq r$.

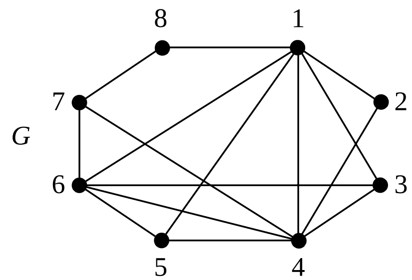
   (f) If $\chi(G) \geq r$, then G contains a subgraph isomorphic to K_r.

6. Determine whether the graphs G and H below are isomorphic. If they are, draw H to look like G. If they are not, explain why not.



G H

7. Determine, with justification, whether the graph G below is Hamiltonian.



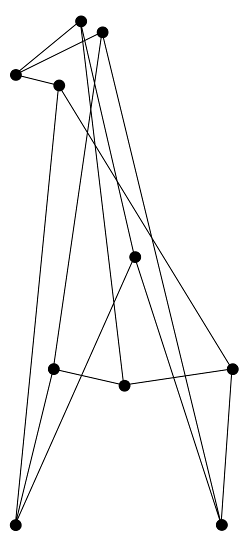
8. (a) Let $H \neq K_n$ be a graph with n vertices and maximum degree $\Delta(H) = n - 1$. Prove that $\chi(H) \leq n - 1$.

   (b) For each $n \geq 3$, give an example of an n-vertex graph $H \neq K_n$ with $\Delta(H) = n - 1$ with the following chromatic numbers, or explain why no such graph exists:

      i. $\chi(H) = n - 1$ ii. $\chi(H) = 2$.

9. Determine the chromatic number of the graph G below. Explain!

10. Determine whether the graph G below is planar. If it is planar, draw a plane representation of G. If it is nonplanar, find a subgraph of G homeomorphic to K_5 or $K_{3,3}$.



11. How many ways are there to divide 44 distinct soccer players into

   (a) four distinct groups of any size for training exercises, at least one player per group?

   (b) four teams, of equal standing, of 11 players each?

12. Determine the number of positive integers x, where $x \leq 99,999$, such that the sum of the digits in x equals 30.

13. How many integers in the set $S = \{n \in \mathbb{Z} : 1 \leq n \leq 10,000\}$ have 3 and 6 and 9 among their digits?

14. How many ways can we place r objects in s boxes if:

(a) The objects are distinguishable, the boxes are distinguishable, and the boxes may be empty?

(b) The objects are distinguishable, the boxes are distinguishable, and the boxes may not be empty?

(c) The objects are indistinguishable, the boxes are distinguishable, and the boxes may be empty?

(d) The objects are indistinguishable, the boxes are distinguishable, and the boxes may not be empty?

(e) The objects are distinguishable, the boxes are indistinguishable, and the boxes may not be empty?

15. How many ways can we partition the set $S = \{1, 2, ..., 6\}$ into at most three subsets? (The order of the subsets is not important.)

16. How many ways are there for a team of 15 rugby players to switch jerseys (with numbers on the back) at half-time so that

    (a) no player plays the two halves in the same jersey?

    (b) no player plays the two halves in the same jersey, and numbers 1, 2, 3 and 4 exchange jerseys among one another, while numbers $5, 6, ..., 15$ exchange jerseys among one another.

17. A permutation matrix is a square matrix ($n \times n$ array) that contains exactly one 1 in each row and each column, with all of the other entries being 0. How many $n \times n$ permutation matrices are there that have all 0's on their main diagonals?

18. Use the principle of inclusion-exclusion to determine $\phi(1001)$, the number of integers less than or equal to $1001 = 7 \times 11 \times 13$ that are relatively prime to 1001.

19. How many 7-digit integers $n \leq 5,000,000$ contain each of the digits $3, 4, 5, 6, 7$ and no others?

20. How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 28$$

    if

    (a) $x_i \geq 1$ for each i?

    (b) $1 \leq x_i \leq 8$ for each i?

21. How many partitions into 4 subsets does the set $\{1, 2, ..., 25\}$ have?