MAT 1341

Fall 2017

Midterm #2, Version #1

1. (1 point)

We are basically asked to do the subspace test on this set of vectors, and asked to determine which part of the subspace test this set of vectors *fails* (or, if it passes all of them, we'll choose F).

So, let's do the subspace test!

$$U = \{(x, y, z, w) \in R^4 \mid xy + zw = 0\}$$

Zero Vector

$$(x, y, z, w) = (0,0,0,0)$$

check restriction:

$$\begin{aligned} xy + zw &= (0)(0) + (0)(0) \\ &= 0 \end{aligned}$$

$\therefore$ U contains a zero vector.

Vector Addition

$$u = (a,b,c,d) \qquad ab + cd = 0 \qquad u \in U$$
$$v = (e,f,g,h) \qquad ef + gh = 0 \qquad v \in U$$

$$\begin{aligned} u + v &= (a,b,c,d) + (e,f,g,h) \\ &= (a+e, b+f, c+g, d+h) \end{aligned}$$

check restriction:

$$\begin{aligned} xy + zw &= (a+e)(b+f) + (c+g)(d+h) \\ &= (ab + af + be + ef) + (cd + ch + dg + gh) \\ &= (ab + cd) + (ef + gh) + af + be + ch + dg \\ &= 0 + 0 + af + be + ch + dg \\ &\neq 0 \end{aligned}$$

$\therefore$ U is not closed under vector addition.

Scalar Multiplication

$$\begin{aligned} &k \in R \\ ku &= k(a,b,c,d) \\ &= (ka, kb, kc, kd) \end{aligned}$$

check restriction:

$$\begin{aligned} xy + zw &= (ka)(kb) + (kc)(kd) \\ &= k^2(ab) + k^2(cd) \\ &= k^2(ab + cd) \\ &= k^2(0) \\ &= 0 \end{aligned}$$

$\therefore$ U is closed under scalar multiplication.

<u>Answer: E</u>

2. (1 point) We can shortcut this and look directly at the restrictions to determine which two are (or are not) subspaces of R^3.

U: Yes. This is a *plane* that passes through 0.

V: No. We have variables *multiplying* each other. This is *non-linear*.

W: Yes. The variables in the vector on the left-side are combined *linearly*.

X: No. Comparing W and X, the variables in the vector on the left-side are combined *non-linearly*.

Answer: B

3. (1 point)

The dimension of a vector space is the number of vectors in the *basis* of the vector space.

A basis has two requirements:

- It must be a spanning set.
- It must contain only linearly independent vectors.

We are told that the vector space *Y* is a subspace of R^{99}, and a linearly independent of 63 vectors. Therefore, *somewhere in the middle there must be a basis*. (The information about it having a spanning set of 75 vectors is a red herring – i.e. a trick! Ignore this information).

Answer: C

4. (1 point)

We can make this question easier by actually making up fake vectors u, v, and w. They don't even have to be traditional vectors, per se.

Let u = milk.

Let v = flour.

Then span{u,v} are all the things you can make when you mix milk and flour. Exciting.

Let w = eggs, something that *cannot* be made by mixing milk and flour (i.e. it is linearly independent).

Which statement is true?

Answer: A (Milk and eggs are independent from each other… in other words, you cannot make milk from eggs, or vice versa. Definitely true.)

5. (3 points)

a) $W = \{ p(x) \in P_2 \mid p(0) = 0 \}$

W is all of the polynomials of 2nd-order (quadratics) where they are equal to 0 when you plug in $x = 0$. We don't need to find *W*, that's just what it is.

Let's do the subspace test on this… thing…

Zero Vector

How to test the zero vector on a polynomial? Well, can I get the polynomial to equal 0 somewhere *guaranteed always*? Yes! If I plug in $x = 0$, I know for sure all of these polynomials will *always equal 0*. That's what the restriction says.

$$p(0) = 0$$

Therefore, *W* contains a zero vector. Yeeeeeah…

Vector Addition

Now I'm going to "create" two polynomials in *W*, called *f* and *g*. They will have the same restriction, such that when we plug $x = 0$ into them, they will equal 0. Then, we can say that "they are part of *W*".

Now we will add these two functions together and see if, when added together, they still meet the restriction. If they do, then W is closed under addition.

The notation for adding functions together is a bit wonky. O_O

$$f(x) \quad f(0) = 0, \quad f \in W$$
$$g(x) \quad g(0) = 0, \quad g \in W$$

check restriction:

$$\begin{aligned}(f+g)(0) &= f(0) + g(0) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

$\therefore$ W is closed under vector addition.

Scalar Multiplication

$$k \in R$$

check restriction:

$$\begin{aligned}(kf)(0) &= kf(0)\\ &= k(0)\\ &= 0\end{aligned}$$

$\therefore$ W is closed under scalar multiplication.

Since W passed all three tests, W is a vector space, and is a *linear subspace of* P_2.

b) A basis for P_2 would be $\{1, x, x^2\}$. W has a restriction on it where, if you plug in $x = 0$, you get an output of 0. If you do that with the basis of P_2 you get $\{1, 0, 0\}$. So that first vector has to go! Therefore, a basis for W is $\{x, x^2\}$.

6. (5 points)

a) There are two ways to prove that *U* is a subspace:

1. Do the subspace test (hard way).
2. Find a basis (easy way).

Let's do the easy way.

First, we'll get a template vector. For a 2x2 matrix, this is the one provided:

$$\begin{bmatrix} a & b \\ a-b & a+c \end{bmatrix}$$

Now we can "factor out" each variable:

$$\begin{bmatrix} a & b \\ a-b & a+c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

… and pop it into a span!

$$\begin{bmatrix} a & b \\ a-b & a+c \end{bmatrix} = span\left\{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Then we can say "*because we have found a spanning set, U is a subspace*."

b) This technique for finding spanning sets will *always find* bases (spanning sets that have linearly independent vectors). So, not only have we found a spanning set, we have actually found a basis in part a.

So uh… yeah… we already answered this in part a.

The basis is $\left\{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

There are three vectors, so dim U = 3.

GIMME MY 2 POINTS.

c) A different one, eh? Hmmm… there are infinite possibilities. A simple way to do this (and a good technique to memorize) is to knock out the original first vector and replace it with the sum of the first two original vectors. This leads to a new basis.

$$\left\{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

Note: You could do this with any single vector, not just the first one. I chose the first one because it was… the first one.

Why does this work?

Back to our recipe/bakery analogy!

Original basis: {flour, milk, eggs}

New basis: {floury milk, milk, eggs}

Still have three independent vectors. The first one is just… weird now. But still independent. It's "pre-mixed".

7. (8 points)

a) Choose a function that is *always positive* to act as a representation for this proof. I will choose $f(x)=e^x$ which is always positive, and meets the restriction $f(x)\geq 0$. Therefore $f(x)=e^x\in X$. If I choose *k = -1*, a scalar, and multiply it by $f(x)=e^x$, I get $f(x)=-e^x$ which is *always negative*. This fails the restriction $f(x)\geq 0$. Therefore, *X* is not closed under scalar multiplication. And because of that, *X* is not a subspace.

False.

b) Hmm… does *U* have a zero vector? In other words, is $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\in U$?

The closest I can get is $\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$ by setting *a = b = c = 0*, but I can't get rid of that *1*. Therefore, *U* does not contain a zero vector. So, *U* is not a subspace.

False.

c) Here is what is required:

The three vectors v_1, v_2, and v_3 must be non-zero, and U must be spanned by them.

Does this guarantee that dim U = 3?

Back to the bakery/recipe analogy (although I don't recommend you write this as your answer on the midterm):

If all three vectors are $v_1 = v_2 = v_3 = eggs$, and $U = span\{eggs, eggs, eggs\}$, this meets the restrictions placed in the question. However, dim U = 1, since there is only one independent "item" in the spanning set. We just have a lot of eggs. So it would be false.

We can then do the same thing with real vectors. Make up a vector!

Let $v_1 = v_2 = v_3 = (1,0)$. Then, $U = span\{(1,0),(1,0),(1,0)\}$. (eggs eggs eggs!). So, dim U = 1.

<u>False.</u>

d) This one's a meanie! Watch out for your trig identities!

$$\begin{aligned} U &= span\{x\sin^2 x, x\cos^2 x, x\} \\ &= span\{v_1, v_2, v_3\} \end{aligned}$$

But…

$$\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ x\cos^2 x + x\sin^2 x &= x \end{aligned}$$

So, we could rewrite one of the vectors in U as a linear combination of the other two. Therefore, these vectors are *linearly dependent*. Therefore, this is not a basis. The statement that dim U = 3 is incorrect.

<u>False.</u>

<u>They're all false. AGGGHGHGH!!!!!</u>

MAT 1341

Fall 2017

Midterm #2, Version #2

1. (1 point)

We are basically asked to do the subspace test on this set of vectors, and asked to determine which part of the subspace test this set of vectors *fails* (or, if it passes all of them, we'll choose F).

So, let's do the subspace test!

$$U = \{(x, y, z, w) \in R^4 \mid xy - zw = 0\}$$

Zero Vector

$$(x, y, z, w) = (0,0,0,0)$$

check restriction:

$$\begin{aligned} xy - zw &= (0)(0) - (0)(0) \\ &= 0 \end{aligned}$$

$\therefore$ U contains a zero vector.

Vector Addition

$$u = (a,b,c,d) \qquad ab - cd = 0 \qquad u \in U$$
$$v = (e,f,g,h) \qquad ef - gh = 0 \qquad v \in U$$

$$\begin{aligned} u + v &= (a,b,c,d) - (e,f,g,h) \\ &= (a-e, b-f, c-g, d-h) \end{aligned}$$

check restriction:

$$\begin{aligned} xy - zw &= (a-e)(b-f) - (c-g)(d-h) \\ &= (ab - af - be + ef) - (cd - ch - dg + gh) \\ &= (ab - cd) + (ef - gh) - af - be + ch + dg \\ &= 0 + 0 - af - be + ch + dg \\ &\neq 0 \end{aligned}$$

$\therefore$ U is not closed under vector addition.

Scalar Multiplication

$$\begin{aligned} &k \in R \\ ku &= k(a,b,c,d) \\ &= (ka, kb, kc, kd) \end{aligned}$$

check restriction:

$$\begin{aligned} xy - zw &= (ka)(kb) - (kc)(kd) \\ &= k^2(ab) - k^2(cd) \\ &= k^2(ab - cd) \\ &= k^2(0) \\ &= 0 \end{aligned}$$

$\therefore$ U is closed under scalar multiplication.

<u>Answer: C</u>

2. (1 point) We can shortcut this and look directly at the restrictions to determine which two are (or are not) subspaces of R^3.

U: Yes. The variables in the vector on the left-side are combined *linearly*.

V: Yes. This is a *plane* that passes through 0.

W: No. We have variables *multiplying* each other. This is *non-linear*.

X: No. The variables in the vector on the left-side are combined *non-linearly*.

Answer: A

3. (1 point)

The dimension of a vector space is the number of vectors in the *basis* of the vector space.

A basis has two requirements:

- It must be a spanning set.
- It must contain only linearly independent vectors.

We are told that the vector space *Y* is a subspace of R^{101}, and a linearly independent of 52 vectors. Therefore, *somewhere in the middle there must be a basis*. (The information about it having a spanning set of 77 vectors is a red herring – i.e. a trick! Ignore this information).

Answer: D

4. (1 point)

We can make this question easier by actually making up fake vectors u, v, and w. They don't even have to be traditional vectors, per se.

Let u = milk.

Let v = flour.

Then span{u,v} are all the things you can make when you mix milk and flour. Exciting.

Let w = eggs, something that *cannot* be made by mixing milk and flour (i.e. it is linearly independent).

Which statement is true?

Answer: C (Flour and eggs are independent from each other… in other words, you cannot make milk from flour, or vice versa. Definitely true.)

5. (3 points)

a) $W = \{p(x) \in P_2 \mid p(0) = 0\}$

W is all of the polynomials of 2nd-order (quadratics) where they are equal to 0 when you plug in *x = 0*. We don't need to find *W*, that's just what it is.

Let's do the subspace test on this… thing…

Zero Vector

How to test the zero vector on a polynomial? Well, can I get the polynomial to equal 0 somewhere *guaranteed always*? Yes! If I plug in *x = 0*, I know for sure all of these polynomials will *always equal 0*. That's what the restriction says.

$p(0) = 0$

Therefore, *W* contains a zero vector. Yeeeeeah…

Vector Addition

Now I'm going to "create" two polynomials in *W*, called *f* and *g*. They will have the same restriction, such that when we plug *x = 0* into them, they will equal 0. Then, we can say that "they are part of *W*".

Now we will add these two functions together and see if, when added together, they still meet the restriction. If they do, then W is closed under addition.

The notation for adding functions together is a bit wonky. O_O

$$f(x) \quad f(0) = 0, \quad f \in W$$
$$g(x) \quad g(0) = 0, \quad g \in W$$

check restriction:

$$\begin{aligned}(f+g)(0) &= f(0) + g(0) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

$\therefore$ W is closed under vector addition.

Scalar Multiplication

$k \in R$

check restriction:

$$(kf)(0) = kf(0) = k(0) = 0$$

$\therefore$ W is closed under scalar multiplication.

Since W passed all three tests, W is a vector space, and is a *linear subspace of P_2.*

b) A basis for P_2 would be $\{1, x, x^2\}$. W has a restriction on it where, if you plug in $x = 0$, you get an output of 0. If you do that with the basis of P_2 you get $\{1, 0, 0\}$. So that first vector has to go! Therefore, a basis for W is $\{x, x^2\}$.

6. (5 points)

a) There are two ways to prove that U is a subspace:

1. Do the subspace test (hard way).
2. Find a basis (easy way).

Let's do the easy way.

First, we'll get a template vector. For a 2x2 matrix, this is the one provided:

$$\begin{bmatrix} a & b \\ a+b & a-c \end{bmatrix}$$

Now we can "factor out" each variable:

$$\begin{bmatrix} a & b \\ a+b & a-c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

… and pop it into a span!

$$\begin{bmatrix} a & b \\ a+b & a-c \end{bmatrix} = span\left\{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right\}$$

Then we can say "*because we have found a spanning set, U is a subspace*."

b) This technique for finding spanning sets will *always find* bases (spanning sets that have linearly independent vectors). So, not only have we found a spanning set, we have actually found a basis in part a.

So uh… yeah… we already answered this in part a.

The basis is $\left\{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right\}$.

There are three vectors, so dim U = 3.

GIMME MY 2 POINTS.

c) A different one, eh? Hmmm… there are infinite possibilities. A simple way to do this (and a good technique to memorize) is to knock out the original first vector and replace it with the sum of the first two original vectors. This leads to a new basis.

$$\left\{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right\}$$

<u>Note:</u> You could do this with any single vector, not just the first one. I chose the first one because it was… the first one.

Why does this work?

Back to our recipe/bakery analogy!

Original basis: {flour, milk, eggs}

New basis: {floury milk, milk, eggs}

Still have three independent vectors. The first one is just… weird now. But still independent. It's "pre-mixed".

7. (8 points)

a) Choose a function that is *always negative* to act as a representation for this proof. I will choose $f(x) = -e^x$ which is always negative, and meets the restriction $f(x) \leq 0$. Therefore $f(x) = -e^x \in X$. If I choose *k = -1*, a scalar, and multiply it by $f(x) = -e^x$, I get $f(x) = e^x$ which is *always positive*. This fails the restriction $f(x) \leq 0$. Therefore, *X* is not closed under scalar multiplication. And because of that, *X* is not a subspace.

False.

b) Hmm… does *U* have a zero vector? In other words, is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in U$?

The closest I can get is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ by setting *a = b = c = 0*, but I can't get rid of that *1*. Therefore, *U* does not contain a zero vector. So, *U* is not a subspace.

False.

c) Here is what is required:

The three vectors v_1, v_2, and v_3 must be non-zero, and U must be spanned by them.

Does this guarantee that dim U = 2?

Back to the bakery/recipe analogy (although I don't recommend you write this as your answer on the midterm):

If all three vectors are $v_1 = v_2 = v_3 = eggs$, and *U = span{eggs, eggs, eggs}*, this meets the restrictions placed in the question. However, dim U = 1, since there is only one independent "item" in the spanning set. We just have a lot of eggs. So it would be false.

We can then do the same thing with real vectors. Make up a vector!

Let $v_1 = v_2 = v_3 = (1,0)$. Then, $U = span\{(1,0),(1,0),(1,0)\}$. (eggs eggs eggs!). So, dim U = 1.

False.

d) This one's a meanie! Watch out for your trig identities!

$$U = span\left\{x^2 \sin^2 x, x^2, x^2 \cos^2 x\right\}$$
$$= span\left\{v_1, v_2, v_3\right\}$$

But…

$$\cos^2 x + \sin^2 x = 1$$
$$x^2 \cos^2 x + x^2 \sin^2 x = x^2$$

So, we could rewrite one of the vectors in *U* as a linear combination of the other two. Therefore, these vectors are *linearly dependent*. Therefore, this is not a basis. The statement that dim U = 3 is incorrect.

False.

They're all false. AGGGHGHGH!!!!!

<u>MAT 1341</u>

<u>Fall 2016</u>

<u>Midterm #2, Version #1</u>

1. We can shortcut this and look directly at the restrictions to determine which two are (or are not) subspaces of R^3.

U: Yes. The variables in the vector on the left-side are combined *linearly*.

V: Yes. The variables in the vector on the left-side are combined *linearly*.

W: No. We have variables *multiplying* each other. This is *non-linear*.

X: Yes. This is a *plane* that passes through 0.

<u>Answer: E</u>

2. Let's analyze each of these individually.

I: Not necessarily. Here are two vectors in R^2: (1,0) and (0,0). However, they do not span all of R^2 because I cannot make every 2D vector with a linear combination of these vectors. For example, I cannot create the vector (1,1) using a combination of these two vectors. So this statement is incorrect.

II: Yes. This is the direction vector of a line in R^2, since it is a 2D vector. The equation of any line is always $r(t) = \text{initial vector} + t(\text{direction vector})$. Since we are not given the formula for the initial vector, it is *implied* that the initial vector is (0,0), the origin. This is a true statement.

III: Let us determine mathematically if we can make the two formulas equal each other by expanding the span and equating it to the linear combination.

$au+b(u+v+w)=au+bu+bv+bw=(a+b)u+bv+bw$

$span\{u,v,w\}=cu+dv+ew$

Equate.

$(a+b)u+bv+bw=cu+dv+ew$

Is this possible?

$a+b=c$
$b=d$
$b=e$

Yes, it is possible to get the two equations to equal each other because their coefficients can equal each other. Therefore, the statement is true.

IV: No. You need at least 4 vectors to span M_{22}.

V: No. You need at least 3 vectors to span R^3.

Answer: C

3. a) We can explain why W is a subspace by finding a basis for W. Only vector spaces (a subspace is a vector space) have bases.

Alrighty… let's find that basis!

$$\begin{aligned}(x,y,z) &= (x,y,x+y)\\ &= x(1,0,1)+y(0,1,1)\\ &= span\{(1,0,1),(0,1,1)\}\end{aligned}$$

Since W has a basis, it must be a subspace.

b) A basis is also a spanning set. Oh hey look… we found a spanning set.

$$W = span\{(1,0,1),(0,1,1)\}$$

c) The technique in part (a) will *always* find a basis (a spanning set with linearly independent vectors). So uh… yeah…

Here's the basis.

This feels like déjà vu.

$$W = span\{(1,0,1),(0,1,1)\}$$

d) W is a plane through the origin with direction vectors (1,0,1) and (0,1,1).

4.

a) To prove that U is a subspace, let's find a basis. I really like doing that… basis basis basis. Remember, if U has a basis, it must be a vector space (and a subspace is a vector space… soooo…).

$$\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= span\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Boom! U has a basis. U is a subspace.

b) …. OK.

$$U = span\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

There are 3 vectors in the basis, therefore *dim U = 3*.

c) NOT ANOTHER ONE. I'll just multiply the vectors in the original basis and… yeah that works. Here's your new basis.

$$U = span\left\{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\right\}$$

Note: You could multiply the original basis by anything. I chose 2. Go crazy on your midterm. Multiply by 9000.

5. a) Choose a function that is *always negative* to act as a representation for this proof. I will choose $f(x) = -x^2$ which is always negative, and meets the restriction $f(x) \leq 0$. Therefore $f(x) = -x^2 \in X$. If I choose *k = -1*, a scalar, and multiply it by $f(x) = -x^2$, I get $f(x) = x^2$ which is *always positive*. This fails the restriction $f(x) \leq 0$. Therefore, *X* is not closed under scalar multiplication. And because of that, *X* is not a subspace.

False.

b) This is ***true***. If $V = span\{v_1, v_2\}$ (in other words, any vector in *V* can be made of a linear combination of v_1 and v_2, or $V = av_1 + bv_2$), then adding another vector to this set won't change that fact. We can then say $V = span\{v_1, v_2, v_3\}$, and generate the statement $V = av_1 + bv_2 + cv_3$. However, we can make this match the original statement $V = av_1 + bv_2$ by setting $c = 0$. So, even adding a new vector v_3 to our set doesn't change the fact that we have a spanning set.

c) Let's try and find a spanning set for this set of vectors. If we can, then W is a subspace.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & -b \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= span\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$$

Since W has a spanning set (which also happens to be a basis), W is a subspace.

True.

d) This is ***false***. The statement given is attempting to say the following:

You have three linearly independent vectors, which then must mean you have a basis (the number of vectors in the basis is the dimension of the vector space).

Recall that a basis has two requirements:

- It is a spanning set
- All of the vectors are linearly independent

Just because we have three vectors that are linearly independent, this doesn't guarantee that the vectors form a spanning set.

Nowhere in the statement does it say the vectors form a spanning set.

So, to prove this statement false, we need to come up with three vectors that are linearly independent, but they don't form a spanning set.

The easiest way to do that is to go to R^4, which requires 4 vectors in its basis. Then, we can come up with 3 linearly independent vectors, but these will not span the vector space.

Here are three vectors in R^4 that are linearly independent:

$$(1,0,0,0),(0,1,0,0),(0,0,1,0)$$

However, since dim $R^4 = 4$, the statement is ***false***.

MAT 1341

Fall 2016

Midterm #2, Version #2

1. We can shortcut this and look directly at the restrictions to determine which two are (or are not) subspaces of R^3.

U: Yes. The variables in the vector on the left-side are combined *linearly*.

V: No. We have variables *multiplying* each other. This is *non-linear*.

W: Yes. This is a *plane* that passes through 0.

X: Yes. The variables in the vector on the left-side are combined *linearly*.

Answer: D

2. Let's analyze each of these individually.

I: Yes. This is the direction vector of a line in R^3, since it is a 3D vector. The equation of any line is always $r(t) = \text{initial vector} + t(\text{direction vector})$. Since we are not given the formula for the initial vector, it is *implied* that the initial vector is (0,0,0), the origin. This is a true statement.

II: Not necessarily. Here are two vectors in R^2: (1,0) and (0,0). However, they do not span all of R^2 because I cannot make every 2D vector with a linear combination of these vectors. For example, I cannot create the vector (1,1) using a combination of these two vectors. So this statement is incorrect.

III: No. You need at least 3 vectors to span R^3.

IV: No. You need at least 4 vectors to span M_{22}.

V: Let us determine mathematically if we can make the two formulas equal each other by expanding the span and equating it to the linear combination.

$au + b(u + v + w) = au + bu + bv + bw = (a + b)u + bv + bw$

$span\{u, v, w\} = cu + dv + ew$

Equate.

$(a + b)u + bv + bw = cu + dv + ew$

Is this possible?

$a + b = c$
$b = d$
$b = e$

Yes, it is possible to get the two equations to equal each other because their coefficients can equal each other. Therefore, the statement is true.

Answer: D

3. a) We can explain why W is a subspace by finding a basis for W. Only vector spaces (a subspace is a vector space) have bases.

Alrighty… let's find that basis!

$$\begin{aligned}(x,y,z)&=(-2y,y,z)\\&=y(-2,1,0)+z(0,0,1)\\&=span\{(-2,1,0),(0,0,1)\}\end{aligned}$$

Since W has a basis, it must be a subspace.

b) A basis is also a spanning set. Oh hey look… we found a spanning set.

$$W=span\{(-2,1,0),(0,0,1)\}$$

c) The technique in part (a) will *always* find a basis (a spanning set with linearly independent vectors). So uh… yeah…

Here's the basis.

This feels like déjà vu.

$$W=span\{(-2,1,0),(0,0,1)\}$$

d) W is a plane through the origin with direction vectors (-2,0,1) and (0,0,1).

4.

a) To prove that U is a subspace, let's find a basis. Remember, if U has a basis, it must be a vector space (and a subspace is a vector space… soooo…). Notice a pattern?

$$\begin{bmatrix} a & b \\ b & a+c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= span\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Boom! U has a basis. U is a subspace.

b) …. OK.

$$U = span\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

There are 3 vectors in the basis, therefore *dim U = 3*.

c) NOT ANOTHER ONE. I'll just multiply the vectors in the original basis and… yeah that works. Here's your new basis.

$$U = span\left\{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\right\}$$

Note: You could multiply the original basis by anything. I chose 2. Go crazy on your midterm. Multiply by 9000.

5. a) Choose a function that is *always greater than or equal to -1* to act as a representation for this proof. I will choose $f(x) = -1$ which meets this restriction since it always equals -1. Therefore $f(x) = -1 \in X$. If I choose *k = 2*, a scalar, and multiply it by $f(x) = -1$, I get $f(x) = -2$ which is *always less than -1*. This fails the restriction. Therefore, *X* is not closed under scalar multiplication. And because of that, *X* is not a subspace.

False.

b) This is ***false***. The vectors in a spanning set are not *necessarily* linearly independent. This is why we have a special term when we do actually meet this condition. A basis is when we have a spanning set *and* the vectors are linearly independent.

So, let's prove this false.

The easiest way would be to create three vectors in R^2. Since dim $R^2 = 2$, any three vectors in R^2 must be linearly *dependent*.

Let $V = R^2$.

Then $V = span\{v_1, v_2, v_3\} = span\{(1,0),(0,1),(1,1)\}$. These three vectors span R^2. However, we can see that $v_1 + v_2 = v_3$. So these vectors are *not* linearly independent. Therefore, the original statement is ***false***.

c) Let's try and find a spanning set for this set of vectors. If we can, then Y is a subspace.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -a & d \end{bmatrix}$$
$$= a\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= span\left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Since Y has a spanning set (which also happens to be a basis), Y is a subspace.

True.

d) This is ***false***. The statement given is attempting to say the following:

You have a spanning set, which then must mean you have a basis (the number of vectors in the basis is the dimension of the vector space).

Recall that a basis has two requirements:

- It is a spanning set
- All of the vectors are linearly independent

Just because we have three vectors that make a spanning set, this doesn't guarantee that the vectors are also linearly independent.

Nowhere in the statement does it say the vectors are linearly independent.

So, to prove this statement false, we need to come up with three vectors that make a spanning set, but are *not* linearly independent.

The easiest way to do that is to go to R^2, which requires 2 vectors in its basis (i.e. $\dim R^2 = 2$). Then, we can come up with 3 vectors which will span the vector space, but they will be linearly *dependent.*

Here are three vectors in R^2 that span R^2:

$$R^2 = span\{(1,0),(0,1),(1,1)\}$$

However, since $(1,0)+(0,1)=(1,1)$, the vectors are not linearly independent, and this is not a basis. Therefore, the statement is ***false***.

MAT 1341

F2015 Test #2

1.

$$U = \{(x, y, z) \in R^3 \mid x - 2y + z = 0\}$$
$$V = \{(x, xy, z) \in R^3 \mid x, y \in R\}$$
$$W = \{(x, y, z) \in R^3 \mid 2x - 5z = 0\}$$
$$X = \{(x + y, y, x - 2y) \in R^3 \mid x, y \in R\}$$

U: Yes, plane = 0.

V: No. In the vector, there is a non-linear component (ie. xy).

W: Yes, plane = 0.

X. Yes. Components in vector are all linear.

Answer: F

2. If Y can be spanned by 96 vectors, then its basis must have $\leq$96 vectors. If Y has a linearly independent set with 71 vectors, then its basis must be $\geq$ 71 vectors.

Recall that a basis is BOTH a spanning set AND linearly independent.

Answer: E

3. Let's do this by process of elimination.

A) No. The three vectors are linearly independent. So, *u* and *w* can't be dependent.

B) No. The three vectors are linearly independent. So, *v* and *w* can't be dependent.

C) No. The question itself states that *u* and *v* are linearly independent.

D) It can't be. The three vectors are linearly independent.

E) It can't be. The three vectors are linearly independent.

F) Sure! Since the three vectors are linearly independent, *w* can't be in the span of *u* and *v*.

Answer: F

4. $W = \{p \in P_2 \mid p(3) = 0\}$

a) To prove that W is a subspace of P_2, we can use the Factor Theorem. This theorem is given in the problem statement.

$$p(x) = (x-a)q(x)$$

where *a* is that guy and *q(x)* is a polynomial of one degree less than what you're dealing with currently (for us, we're looking at P_2, so one degree less is a linear function).

So,

$$\begin{aligned} p(x) &= (x-3)(ax+b) \\ &= ax^2 - 3ax + bx - 3b \\ &= a(x^2-3x) + b(x-3) \\ &= span\{x^2-3x, x-3\} \end{aligned}$$

Oh hey… that's the same spanning set that they gave us in the problem statement.

Since *W* has a spanning set, it is a subspace of P_2.

b) $W = span\{x-3, x^2-3x\}$ (given in (a)). Since *W* has a spanning set, it is a subspace of P_2. I JUST SAID THAT.

c) We will determine whether the "vectors" in the given spanning set are linearly independent. If they are, then the spanning set is already a basis.

$$W = span\left\{x-3, x^2-3x\right\}$$
$$= a\left(x-3\right)+b\left(x^2-3x\right)$$

$$a\left(x-3\right)+b\left(x^2-3x\right)=0$$

$$\boxed{x=0}$$
$$-3a=0$$
$$a=0$$

$$\boxed{x=1}$$
$$-2b=0$$
$$b=0$$

The only solution is the trivial solution, which indicates that these two "vectors" are linearly independent. Therefore, this is already a basis.

Thus, a basis for W is $\left\{x-3, x^2-3x\right\}$.

d) dim W = 2 (there are two vectors in the basis)

5. $U = \left\{ \begin{bmatrix} a & b \\ a+b & c \end{bmatrix} \in M_{22} \mid a,b,c \in R \right\}$

a) The theorem is easier. If we can find a spanning set for *U*, then it is a subspace of M_{22}.

$$\begin{bmatrix} a & b \\ a+b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= span\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Hey look, a spanning set! Since *U* has a spanning set, it is a subspace of M_{22}.

b) Oh hey… I done did dat already. $U = span\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

This set of "vectors" is already linearly independent***, and it is a spanning set. Therefore it is a basis. There are three vectors in this basis, so dim U = 3.

***To prove this, we can state that

$$a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} a & b \\ a+b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$a = 0$$
$$b = 0$$
$$c = 0$$

d) Any scalar multiple of a basis is also a basis. So, $U = span\left\{ \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right\}$ is a basis.

Or if you wanna be #extra, $U = span\left\{ \begin{bmatrix} 9000 & 0 \\ 9000 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 9000 \\ 9000 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 9000 \end{bmatrix} \right\}$ is also a basis.

6. a) $X = \{ f \in F(R) \mid f(x) \le -1 \quad \text{for all } x \in R \}$ is a subspace of $F(R)$

HAHA NOPE!

We can prove this by checking whether or not it's closed under scalar multiplication (hint: it's not).

$k = -1$
$f(x) = 5 \in X$
$kf(x) = -5 \notin X$

Since X is not closed under scalar multiplication, it is not a subspace of F(R).

Answer: False

b) This is from a theorem that we all slept through in class that states that "any subset of a linearly independent set is also linearly independent".

The proof is as follows:

$av_1 + bv_2 + cv_3 = 0$ is linearly independent implies that a = b = c = 0 is the only solution.

Then, we can sub in $c = 0$.

$av_1 + bv_2 + 0v_3 = 0$ (recall that a = b = c = 0 is still true)

Which means that v_3 doesn't even have to be there. This statement could be rewritten as

$av_1 + bv_2 = 0$

So a = b = 0 is still true. Because of that, v_1 and v_2 are linearly independent.

Answer: True

c) That's true. Let's prove it by finding a spanning set for this set of wannabe vectors.

$$\begin{bmatrix} a & a \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= span\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Since this set of "vectors" has a spanning set, it is a subspace of M_{22}.

We have just found a basis (the vectors in the spanning set are linearly independent***). There are 3 vectors in it. So, dim = 3.

***To prove this, we can state that

$$a\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & a \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a = 0$$

$$b = 0$$

$$c = 0$$

<u>Answer: True</u>

d) That can't be true. We need to find four non-zero vectors that form a spanning set (NOT a basis) of some vector space. Then, prove that the dimension of the vector space is NOT always going to be 4.

Here's an easy counterexample:

Make all the vectors the same (the question didn't say they couldn't be!).

$U = span\{v_1, v_2, v_3, v_4\} = span\{(1,1),(1,1),(1,1),(1,1)\}$.

The basis of this vector space would be $U = span\{(1,1)\}$ which has dim U = 1.

<u>Answer: False</u>

MAT 1341

W2015 Test #2

1. $U = \{(x, y, z, w) \in R^4 \mid xyz = 0\}$

Subspace Test.

Zero vector?

$(x, y, z, w) = (0, 0, 0, 0)$

Check restriction.

$xyz = (0)(0)(0) = 0$

Yes, U has a zero vector.

Closed under addition?

$u = (a, b, c, d), \quad abc = 0, \quad u \in U$
$v = (e, f, g, h), \quad efg = 0 \quad v \in U$

$u + v = (a, b, c, d) + (e, f, g, h) = (a + e, b + f, c + g, d + h)$

Check restriction.

$$\begin{aligned}(a+e)(b+f)(c+g) &= (a+e)(bc + bg + cf + fg) \\ &= abc + abg + acf + afg + bce + beg + cef + efg \\ &= 0 + abg + acf + afg + bce + beg + cef + 0 \\ &= abg + acf + afg + bce + beg + cef \\ &\neq 0\end{aligned}$$

U is not closed under vector addition.

Closed under scalar multiplication?

$k \in R$

$ku = k(a,b,c,d) = (ka,kb,kc,kd)$

Check restriction.

$$\begin{aligned}(ka)(kb)(kc) &= k^3(abc) \\ &= k^3(0) \\ &= 0\end{aligned}$$

U is closed under scalar multiplication.

Answer: D

2.

$U = \{(x,y,z) \mid 2x - y + 3z = 0\}$

$V = \{(x,y,z) \mid xy = 0\}$

$W = \{(x,y,z) \mid 2x = 5z\}$

$X = \{(x,y,z) \mid x = y + 3 = 7z\}$

U: Yes, written as a plane = 0.

V: No, non linear.

W: Yes, can be rearranged to $2x - 5z = 0$ which is a plane = 0.

X: No, can be rearranged to $x - y - 7z = -3$ which is a plane = -3.

Answer: D

3.

$S = \{ f \in F[R] \mid f(0) = 1 \}$
$T = \{ f \in F[R] \mid f(1) = 0 \}$
$U = \{ f \in F[R] \mid f(0)f(1) = 0 \}$
$V = \{ f \in F[R] \mid f(x) = f(-x), \quad \forall x \in R \}$

S: No, does not contain a zero vector (since f(0) = 1, not = 0).

T: Yes, written as a "plane" (ie. f(1) = 0).

U: No, non-linear (ie. f(0)*f(1)).

V: Yes, can be rewritten as f(x) – f(-x) = 0, which is a "plane".

Answer: D

4. $W = \{ (x, y, z) \in R^3 \mid x - y + 3z = 0 \}$

a) To prove that W is a subspace of R^3, we can find a spanning set.

$x - y + 3z = 0$
$x = y - 3z$

$$\begin{aligned}(x, y, z) &= (y - 3z, y, z)\\ &= y(1,1,0) + z(-3,0,1)\\ &= span\{(1,1,0), (-3,0,1)\}\end{aligned}$$

Since *W* has a spanning set, it is a subspace of R^3.

b) Hey look, I already did that.

$W = span\{(1,1,0), (-3,0,1)\}$

c) W is a plane in R^3 that passes through the origin with direction vectors (1,1,0) and (-3,0,1).

5. $U = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \in M_{22} \mid a, b \in R \right\}$

a) The theorem is easier. If we can find a spanning set for *U*, then it is a subspace of M_{22}.

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} = a\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= span\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

Hey look, a spanning set! Since *U* has a spanning set, it is a subspace of M_{22}.

b) Oh hey… I done did dat already.

$$U = span\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

c) I need a matrix that is a 2x2 matrix, but it isn't in the subspace U. Cool… There are a lot of possibilities. Here's one.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

6. a) $X = \{ f \in F(R) \mid f(x) \leq 0 \quad \text{for all } x \in R \}$ is a subspace of $F(R)$

JOKE'S ON YOU, MATH, IT'S NOT.

We can prove this by checking whether or not it's closed under scalar multiplication (hint: it's not).

$k = -1$
$f(x) \leq 0$
$kf(x) = -f(x) \geq 0$

Since *X* is not closed under scalar multiplication, it is not a subspace of F(R).

<u>Answer: False</u>

b) I have a suspicion that this is false, so I'm going to try to find a counterexample. I need to find a subspace *U* that contains *u* – *v*, both doesn't contain the individual vectors *u* and *v*, which are R^2 vectors.

I'll make up a really simple vector space, say $U = \{(x,0) \in R^2 \mid x \in R\}$. Then, if I make $u = (1,1)$ and $v = (1,1)$, the result $u - v = (0,0)$ which is contained in *U* (if *x = 0*, the vector (0,0) is a part of *U*). HOWEVER, both *u* and *v* are NOT in *U*. So, we have found a counterexample!

<u>Answer: False</u>

c) That's true. Let's prove it by finding a spanning set for this set of wannabe vectors.

$$\begin{bmatrix} a & a \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= span\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Since this set of "vectors" has a spanning set, it is a subspace of M_{22}.

<u>Answer: True</u>

d) This question pops up a lot as a T/F on the midterm/final exam. The proof is pretty standard. It's a good one to go through and ~~learn~~ *cough* memorize.

$$span\{u, v\} = span\{u, u+v\}$$

We will expand both sides of the span.

$$span\{u, v\} = au + bv$$

$$span\{u, u+v\} = cu + d(u+v)$$
$$= cu + du + dv$$
$$= u(c+d) + dv$$

Now we will set these expanded spans (span span span) equal to each other.

$$au + bv = u(c+d) + dv$$

$$u : a = c + d$$
$$v : b = d$$

We basically just need to confirm whether the above two equations (with the a,b,c,d) allow us to plug *any* value into the left-side, and then produce that value on the right-side.

For example, if I wanted (a,b) to be (2,3), is that possible with (c,d)? Yes. I could use (c,d) = (-1,3). The value of d is set by b, and c is used to help make the value of a.

This is true for any combination of (a,b) that we can come up with. I always do this little check with a pair of numbers to see if it works, and then if it does, I'm fairly confident it'll work for any real numbers (not really a good idea to write down on your exam… but a good idea to check if you're going in the right direction for a proof!).

Answer: True

MAT 1341

F2014 Test #2

1. $X = \{(a,b,c) \in R^3 \mid bc = 0\}$

Subspace Test.

Zero vector?

$(a,b,c) = (0,0,0)$

Check restriction.

$bc = (0)(0) = 0$

Yes, X has a zero vector.

Closed under addition?

$$u = (a,b,c), \quad bc = 0, \quad u \in U$$
$$v = (d,e,f), \quad ef = 0 \quad v \in U$$

$$u + v = (a,b,c) + (d,e,f) = (a+d, b+e, c+f)$$

Check restriction.

$$\begin{aligned}(b+e)(c+f) &= bc + ce + bf + ef \\ &= 0 + ce + bf + 0 \\ &= ce + bf \\ &\neq 0\end{aligned}$$

X is not closed under vector addition.

Closed under scalar multiplication?

$k \in R$

$ku = k(a,b,c) = (ka,kb,kc)$

Check restriction.

$$
\begin{aligned}
(kb)(kc) &= k^2(bc) \\
&= k^2(0) \\
&= 0
\end{aligned}
$$

X is closed under scalar multiplication.

<u>Answer: C</u>

2.

(1) $\{(x,x+y,x+2y) \in R^3 \mid x,y \in R\}$

(2) $\{(x,y,z) \in R^3 \mid x-2=y-3=z\}$

(3) $\{(x,y,z) \in R^3 \mid xyz=0\}$

(4) $\{(x,y,z) \in R^3 \mid x-y-z=0\}$

1: Yes. Components in vector are all linear.

2: No, rearranges to $x - y - z = 1$.

3: No, non-linear component (ie. xyz).

4. Yes. Plane = 0.

<u>Answer: E</u>

3.

A) No. Doesn't have a zero vector.

B) Yes.

C) No. Non-linear component (ie. *ab = 1*).

D) No. Restriction states that *a,b,c* must be integers. Vector spaces exist in all real space.

E) No. Non-linear component (ie. *cd = 0*).

F) Nope. I found one.

Answer: B

4. $W = \{(x, y, z) \in R^3 \mid x + 5y - 4z = 0\}$

a) To prove that W is a subspace, we will find a spanning set.

$$x + 5y - 4z = 0$$
$$x = -5y + 4z$$
$$(x, y, z) = (-5y + 4z, y, z)$$
$$= y(-5,1,0) + z(4,0,1)$$
$$= span\{(-5,1,0),(4,0,1)\}$$

Since *W* has a spanning set, it is a subspace of R^3.

b) Where did you go? I miss you.

c) Oh hey I did that already.

$$W = span\{(-5,1,0),(4,0,1)\}$$

d) W is a plane in R^3 that passes through the origin with direction vectors (-5,1,0) and (4,0,1).

5. $U = \left\{ \begin{bmatrix} 0 & a \\ b & 2a \end{bmatrix} \in M_{22} \mid a, b \in R \right\}$

a) The theorem is easier. If we can find a spanning set for *U*, then it is a subspace of M_{22}.

$$\begin{bmatrix} 0 & a \\ b & 2a \end{bmatrix} = a\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$= span\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

Hey look, a spanning set! Since *U* has a spanning set, it is a subspace of M_{22}.

b) Oh hey… I done did dat already. $U = span\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

c) We need a 2x2 matrix that is… a 2x2 matrix, but *isn't* in our subspace *U*. There's a bunch of them (an infinite number, in fact, har har har SHUT UP DAD). Here's one:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

6. a) $X = \{f \in F(R) \mid f(2) \le 0 \quad \text{for all } x \in R\}$ is a subspace of $F(R)$

Nopers.

We can prove this by checking whether or not it's closed under scalar multiplication (hint: it's not).

$k = -1$
$f(2) \le 0 \in X$
$kf(x) = -f(2) \ge 0 \notin X$

Since *X* is not closed under scalar multiplication, it is not a subspace of F(R).

<u>Answer: False</u>

b) I have a suspicion that this is false, so I'm going to try to find a counterexample. I need to find a subspace *X* that contains *v + 2w*, both doesn't contain the individual vectors *v* and *w*, which are R^2 vectors.

I'll make up a really simple vector space, say $X = \{(x,0) \in R^2 \mid x \in R\}$. Then, if I make $v = (2,2)$ and $w = (-1,-1)$, the result $v + 2w = (0,0)$ which is contained in *X* (if *x = 0*, the vector (0,0) is a part of *X*). HOWEVER, both *v* and *w* are NOT in *X*. So, we have found a counterexample!

<u>Answer: False</u>

c) That's true. Let's prove it by finding a spanning set for this set of wannabe vectors.

$$\begin{bmatrix} a & b \\ a & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= span\left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Since this set of "vectors" has a spanning set, it is a subspace of M_{22}.

<u>Answer: True</u>

d) This question pops up a lot as a T/F on the midterm/final exam. The proof is pretty standard. It's a good one to go through.

$$span\{v, w\} = span\{v + w, v\}$$

We will expand both sides of the span.

$$span\{v, w\} = av + bw$$

$$\begin{aligned} span\{v + w, v\} &= c(v + w) + dv \\ &= cv + cw + dv \\ &= v(c + d) + cw \end{aligned}$$

Now we will set these expanded spans equal to each other.

$$av + bw = v(c + d) + cw$$

$$\begin{aligned} v &: a = c + d \\ w &: b = c \end{aligned}$$

We basically just need to confirm whether the above two equations (with the a,b,c,d) allow us to plug *any* value into the left-side, and then produce that value on the right-side.

For example, if I wanted (a,b) to be (2,3), is that possible with (c,d)? Yes. I could use (c,d) = (3,-1). The value of c is set by b, and d is used to help make the value of a.

This is true for any combination of (a,b) that we can come up with. I always do this little check with a pair of numbers to see if it works, and then if it does, I'm fairly confident it'll work for any real numbers (not really a good idea to write down on your exam… but a good idea to check if you're going in the right direction for a proof!).

Answer: True