EXAMPLE 1-1:

a)

$a(1,3,1)+b(0,0,2)+c(1,1,1)=(0,0,0)$

$$\begin{aligned} a+c&=0\\ 3a+c&=0\\ a+2b+c&=0 \end{aligned}$$

$$\left[\begin{array}{ccc|c}1&0&1&0\\3&0&1&0\\1&2&1&0\end{array}\right]=\left[\begin{array}{ccc|c}1&0&1&0\\0&0&-2&0\\0&2&0&0\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}1&0&0&0\\0&0&1&0\\0&1&0&0\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

$\therefore$ linearly independent

b)

$a(3,2)+b(1,1)+c(2,3)=(0,0)$

$$\begin{aligned} 3a+b+2c&=0\\ 2a+b+3c&=0 \end{aligned}$$

$$\left[\begin{array}{ccc|c}3&1&2&0\\2&1&3&0\end{array}\right]=\left[\begin{array}{ccc|c}1&1/3&2/3&0\\0&1/3&5/3&0\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}1&1/3&2/3&0\\0&1&5&0\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&5&0\end{array}\right]$$

$\therefore$ linearly dependent

<u>EXAMPLE 1-2:</u>

$$ax + b\left(x^3 + 1\right) + c\left(-2x\right) = 0$$

$$\boxed{x = 0}$$
$$b = 0$$

$$\boxed{x = -1}$$
$$-a + 2c = 0$$

$$\boxed{x = 2}$$
$$2a - 4c = 0$$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore$ linearly dependent

1.4 Linear Independence: Your Turn!

Solutions:

a)

$a(1,0,2,1)+b(1,2,0,1)+c(-1,0,0,1)=(0,0,0,0)$

$$\begin{aligned} a+b-c&=0\\ 2b&=0\\ 2a&=0\\ a+b+c&=0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0\\ 0 & 2 & 0 & 0\\ 2 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{-2R_1+R_3\\ -R_1+R_4}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0\\ 0 & 2 & 0 & 0\\ 0 & -2 & 2 & 0\\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2/2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & -2 & 2 & 0\\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{-R_2+R_1\\ 2R_2+R_3}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3/2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_3+R_1\\ -2R_3+R_4}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{array}\right]$$

$a=b=c=0$

$\therefore$ linearly independent

b)

$$a\left(3\sin^2 x\right)+b\left(4\cos^2 x\right)+c=0$$

Usually, I would choose $x = 0$, $\pi/2$, and π for a trig question. However, in this case, $x = \pi/2$ and $x = \pi$ give the same equations. Using them would give the wrong conclusion (as discussed in the instructions earlier). Instead, I will use $x = \pi/2$ and $x = \pi/4$, which do NOT give the same equations, and get the correct answer.

$\boxed{x=0}$

$$4b+c=0$$

$\boxed{x=\pi/4}$

$$\frac{3}{2}a+2b+c=0$$

$\boxed{x=\pi/2}$

$$3a+c=0$$

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 0 \\ 3/2 & 2 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{swap\ R_1,R_2}\left[\begin{array}{ccc|c} 3/2 & 2 & 1 & 0 \\ 0 & 4 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1/(3/2)}\left[\begin{array}{ccc|c} \lfloor 1 & 4/3 & 2/3 & 0 \\ 0 & 4 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-3R_1+R_3}\left[\begin{array}{ccc|c} \lfloor 1 & 4/3 & 2/3 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & -4 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2/4}\left[\begin{array}{ccc|c} \lfloor 1 & 4/3 & 2/3 & 0 \\ 0 & \lfloor 1 & 1/4 & 0 \\ 0 & -4 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[4R_2+R_3]{-4/3R_2+R_1}\left[\begin{array}{ccc|c} \lfloor 1 & 0 & \boxed{1/3} & 0 \\ 0 & \lfloor 1 & \boxed{1/4} & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

There is a parameter!

$\therefore$ linearly dependent

EXAMPLE 2-1:

Zero Vector

$$(x, y, z) = (0,0,0)$$

check restriction:

$$\begin{aligned} xy - z &= (0)(0) - 0 \\ &= 0 \end{aligned}$$

$\therefore$ T contains a zero vector.

Vector Addition

$$u = (a,b,c) \qquad ab - c = 0 \qquad u \in T$$
$$v = (d,e,f) \qquad de - f = 0 \qquad v \in T$$

$$\begin{aligned} u + v &= (a,b,c) + (d,e,f) \\ &= (a+d, b+e, c+f) \end{aligned}$$

check restriction:

$$\begin{aligned} xy - z &= (a+d)(b+e) - (c+f) \\ &= ab + bd + ae + de - c - f \\ &= (ab - c) + (de - f) + bd + ae \\ &= 0 + 0 + bd + ae \\ &\neq 0 \end{aligned}$$

$\therefore$ T is not closed under vector addition.

Since T failed one of the tests, T is not a vector space.

EXAMPLE 2-2:

We can shortcut this and look directly at the restrictions to determine which two are (or are not) subspaces of F.

a) No. Doesn't equal 0.
b) No. Two functions multiplying each other.
c) Yes. Can be rearranged to equal 0.
d) Yes. Is rearranged to equal 0.

2.4 An Introduction to Vector Spaces: Your Turn!

Solutions:

a)

Zero Vector

$f(x) \leq 0$

$\therefore$ A contains a zero vector.

Vector Addition

$$g(x): \quad g(x) \leq 0 \quad g \in A$$
$$h(x): \quad h(x) \leq 0 \quad h \in A$$

$$g(x) + h(x) = (g+h)(x)$$

check restriction:

$$\underbrace{g(x)}_{\leq 0} + \underbrace{h(x)}_{\leq 0} \leq 0$$

$\therefore$ A is closed under vector addition.

Scalar Multiplication

$$k = -1$$

check restriction:

$$kg(x) = (-1)\underbrace{g(x)}_{\leq 0}$$
$$\geq 0$$

$\therefore$ A is not closed under scalar multiplication.

Since A failed one of the tests, A is not a vector space.

b)

Zero Vector

$$(x, y) = (0,0)$$

check restriction:

$$3xy = 3(0)(0)$$
$$= 0$$

$\therefore$ B contains a zero vector.

Vector Addition

$$u = (a,b) \qquad 3ab = 0 \qquad u \in B$$
$$v = (c,d) \qquad 3cd = 0 \qquad v \in B$$

$$u + v = (a,b) + (c,d)$$
$$= (a + c, b + d)$$

check restriction:

$$3xy = (a + c)(b + d)$$
$$= 3ab + 3bc + 3ad + 3cd$$
$$= 0 + 3bc + 3ad + 0$$
$$= 3bc + 3ad$$
$$\neq 0$$

$\therefore$ B is not closed under vector addition.

<u>Since B failed one of the tests, B is not a vector space.</u>

c)

Zero Vector

$f(-2x) = 0$

$\therefore$ C contains a zero vector.

Vector Addition

$$g(-2x): \quad g(-2x) = 0 \quad g \in C$$
$$h(-2x): \quad h(-2x) = 0 \quad h \in C$$

$g(-2x) + h(-2x) = (g+h)(-2x)$

check restriction:

$$\begin{aligned} g(-2x) + h(-2x) &= 0 + 0 \\ &= 0 \end{aligned}$$

$\therefore$ C is closed under vector addition.

Scalar Multiplication

$$k \in R$$
$$kg(-2x) = (kg)(-2x)$$

check restriction:

$$\begin{aligned} kg(-2x) &= k(0) \\ &= 0 \end{aligned}$$

$\therefore$ C is closed under scalar multiplication.

<u>Since C passed all of the tests, C is a vector space.</u>

d)

Zero Vector

$$(x, y, z, w) = (0,0,0,0)$$

check restriction:

$x = 0, y = 0, z = 0, w = 0$

$\therefore$ D contains a zero vector.

Vector Addition

$u = (a,b,c,d) \qquad a,b,c = 0, d \in R \qquad u \in D$

$v = (e,f,g,h) \qquad e,f,g = 0, h \in R \qquad v \in D$

Alternatively, can write

$u = (0,0,0,a) \qquad a \in R \qquad u \in D$

$v = (0,0,0,b) \qquad b \in R \qquad v \in D$

I will use this form in the solution. Either is correct.

$$\begin{aligned} u + v &= (0,0,0,a) + (0,0,0,b) \\ &= (0+0, 0+0, 0+0, a+b) \\ &= (0,0,0,a+b) \end{aligned}$$

check restriction:

$x = 0$

$y = 0$

$z = 0$

$w = \underbrace{a}_{R} + \underbrace{b}_{R} \in R$

$\therefore$ D is closed under vector addition.

Scalar Multiplication

$$k \in R$$
$$ku = k(0,0,0,a)$$
$$= (0,0,0,ka)$$

check restriction:

$$x = 0$$
$$y = 0$$
$$z = 0$$
$$w = \underbrace{ka}_{R \times R = R} \in R$$

$\therefore$ D is closed under scalar multiplication.

Since D passed all of the tests, D is a vector space.

EXAMPLE 3-1:

$$\begin{aligned}(x,y,z) &= (x,y,x+2y)\\ &= x(1,0,1)+y(0,1,2)\\ &= span\{(1,0,1),(0,1,2)\}\end{aligned}$$

Since U has a spanning set, U is a vector space.

I JUST SAID.

I FOUND.

A SPANNING SET.

And it's also a basis. Cuz I haz teh awesomez.

dim $U = 2$

$R^3 = span\{(1,0,1),(0,1,2),(1,0,0)\}$

EXAMPLE 3-2:

$$a(1,0,0)+b(2,0,1)+c(1,1,1)+d(0,0,2)=(0,0,0)$$

$$\begin{aligned} a+2b+c&=0 \\ c&=0 \\ b+c+2d&=0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} \underline{1} & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{swap\ R_2,R_3} \left[\begin{array}{cccc|c} \underline{1} & 2 & 0 & 0 & 0 \\ 0 & \underline{1} & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{cccc|c} \underline{1} & 0 & -2 & -4 & 0 \\ 0 & \underline{1} & 1 & 2 & 0 \\ 0 & 0 & \underline{1} & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-R_3+R_2]{2R_3+R_1} \left[\begin{array}{cccc|c} \underline{1} & 0 & 0 & \boxed{-4} & 0 \\ 0 & \underline{1} & 0 & \boxed{2} & 0 \\ 0 & 0 & \underline{1} & \boxed{0} & 0 \end{array}\right]$$

Since *d* is a parameter, we will eliminate its corresponding vector from the spanning set to form a basis.

$$\begin{aligned} V &= span\{(1,0,0),(2,0,1),(1,1,1),\cancel{(0,0,2)}\} \\ &= span\{(1,0,0),(2,0,1),(1,1,1)\} \end{aligned}$$

$$\dim V = 3$$

3.2 More Vector Space Stuff: Your Turn!

Solutions:

a)

i) We can do this the hard way or the easy way. The hard way is to do the *subspace test.* If you want to do that, you can. It is not incorrect. However, we can shortcut this whole thing and do something muuuuuch faster!

If we can find a spanning set for U, then this will prove that U is a vector space, because only vector spaces can have spanning sets. So… I'MA DO DIS.

$$U = \{(x, y, z) \in R^3 \mid x - 4y - 2z = 0\}$$

$$x - 4y - 2z = 0$$
$$x = 4y + 2z$$

$$\begin{aligned}(x, y, z) &= (4y + 2z, y, z) \\ &= y(4,1,0) + z(2,0,1) \\ &= span\{(4,1,0),(2,0,1)\}\end{aligned}$$

Since U has a spanning set, U is a vector space.

ii) I just… I just found the… I just found the spanning set…

$U = span\{(4,1,0),(2,0,1)\}$ 

iii) This spanning set is actually a basis, so uh… here it is again.

$U = span\{(4,1,0),(2,0,1)\}$

iv) The basis has two vectors in it, so *dim U = 2*.

v) Many more answers are possible.

$R^3 = span\{(4,1,0),(2,0,1),(0,0,1)\}$ or $R^3 = span\{(4,1,0),(2,0,1),(0,1,0)\}$ are both valid given the instructions I have provided.

b)

$$a(1,1,2,3)+b(0,2,1,1)+c(-4,-4,-8,-12)+d(2,0,3,5)=(0,0,0,0)$$

$$\begin{aligned} a-4c+2d&=0\\ a+2b-4c&=0\\ 2a+b-8c+3d&=0\\ 3a+b-12c+5d&=0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} \underline{1} & 0 & -4 & 2 & 0\\ 1 & 2 & -4 & 0 & 0\\ 2 & 1 & -8 & 3 & 0\\ 3 & 1 & -12 & 5 & 0 \end{array}\right]$$

$$\xrightarrow[-3R_1+R_4]{\substack{-R_1+R_2\\ -2R_1+R_3}} \left[\begin{array}{cccc|c} \underline{1} & 0 & -4 & 2 & 0\\ 0 & 2 & 0 & -2 & 0\\ 0 & 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2/2} \left[\begin{array}{cccc|c} \underline{1} & 0 & -4 & 2 & 0\\ 0 & \underline{1} & 0 & -1 & 0\\ 0 & 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[-R_2+R_4]{-R_2+R_3} \left[\begin{array}{cccc|c} \underline{1} & 0 & \boxed{-4} & \boxed{2} & 0\\ 0 & \underline{1} & \boxed{0} & \boxed{-1} & 0\\ 0 & 0 & \boxed{0} & \boxed{0} & 0\\ 0 & 0 & \boxed{0} & \boxed{0} & 0 \end{array}\right]$$

Since *c* and *d* are parameters, we will eliminate their corresponding vectors from the spanning set to form the basis.

$$V = span\{(1,1,2,3),(0,2,1,1),\cancel{(-4,-4,-8,-12)},\cancel{(2,0,3,5)}\}$$
$$= span\{(1,1,2,3),(0,2,1,1)\}$$

This is a basis for V. Since this is the basis, and it has two vectors, $\dim V = 2$.