EXAMPLE 1-1:

Matrix form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right]$$

EXAMPLE 1-2:

Matrix form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & 1 & 1 & 4 \\ 3 & 1 & 1 & 6 \end{array}\right]$$

Gauss time!

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 5 \\ 2 & 1 & 1 & 4 \\ 3 & 1 & 1 & 6 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 5 \\ 0 & -3 & -1 & -6 \\ 0 & -5 & -2 & -9 \end{array}\right]$$

$$\xrightarrow{R_2/-3} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 5 \\ 0 & \boxed{1} & 1/3 & 2 \\ 0 & -5 & -2 & -9 \end{array}\right]$$

$$\xrightarrow[5R_2+R_3]{-2R_2+R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1/3 & 1 \\ 0 & \boxed{1} & 1/3 & 2 \\ 0 & 0 & -1/3 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1/3 & 1 \\ 0 & \boxed{1} & 1/3 & 2 \\ 0 & 0 & \boxed{1} & -3 \end{array}\right]$$

$$\xrightarrow[-1/3R_3+R_2]{-1/3R_3+R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{1} & -3 \end{array}\right]$$

Solution type: Unique solutions.

$x = 2$
$y = 3$ parametric form
$z = -3$

$(x, y, z) = (2, 3, -3)$ vector form

EXAMPLE 1-3:

Matrix form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 2 \\ 2 & 6 & 10 & 4 \end{array}\right]$$

Gauss time!

$$\left[\begin{array}{ccc|c} \underline{1} & 2 & 3 & 1 \\ 0 & 2 & 4 & 2 \\ 2 & 6 & 10 & 4 \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|c} \underline{1} & 2 & 3 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_2/2} \left[\begin{array}{ccc|c} \underline{1} & 2 & 3 & 1 \\ 0 & \underline{1} & 2 & 1 \\ 0 & 2 & 4 & 2 \end{array}\right]$$

$$\xrightarrow[-2R_2+R_3]{-2R_2+R_1} \left[\begin{array}{ccc|c} \underline{1} & 0 & \boxed{-1} & -1 \\ 0 & \underline{1} & \boxed{2} & 1 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

Solution type: Infinite solutions.

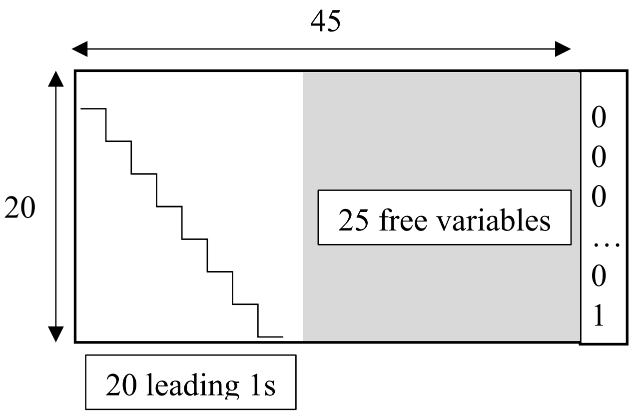
$x_1 = -1 + t$

$x_2 = 1 - 2t$ parametric form

$x_3 = t$

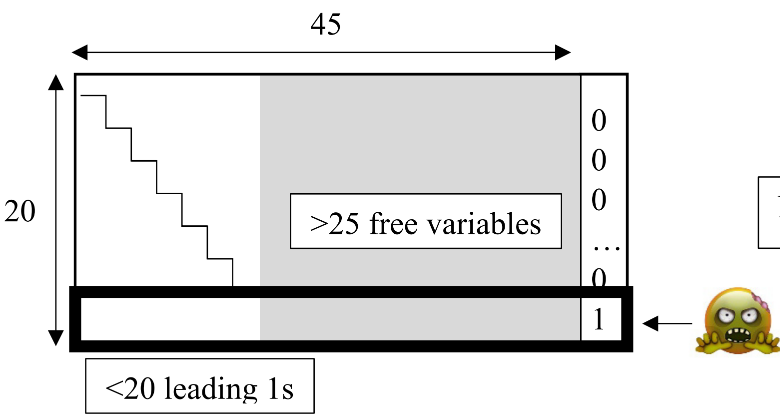
$(x_1, x_2, x_3) = (-1, 1, 0) + t(1, -2, 1)$ vector form

EXAMPLE 1-4:



Unique solution is *impossible*.

Infinite solutions is *possible*.



No solutions is *possible*.

EXAMPLE 1-5:

Gauss!

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & a-3 & b-2 \end{array}\right]$$

$$\xrightarrow[-2R_2+R_3]{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-4 \end{array}\right] \longleftarrow \boxed{\text{Stop here!}}$$

No solutions:

Need a zombie row. This happens if $a = 5, b \neq 4$.

Unique solution:

Need a non-zombie, non-dead row. This happens if $a \neq 5, b$ can be any real number.

Infinite solutions:

Need a dead row. This happens if $a = 5, b = 4$.

EXAMPLE 1-6:

$$\left[\begin{array}{cccc|c} \underline{1} & 2 & 1 & 4 & 5 \\ 1 & 1 & 1 & 0 & 2 \\ 3 & 6 & 3 & 12 & 15 \\ 2 & 2 & 2 & 0 & 3 \end{array}\right] \xrightarrow[-2R_1+R_3]{\substack{-R_1+R_2 \\ -3R_1+R_2}} \left[\begin{array}{cccc|c} \underline{1} & 2 & 1 & 4 & 5 \\ 0 & -1 & 0 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -8 & -7 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{cccc|c} \underline{1} & 2 & 1 & 4 & 5 \\ 0 & \underline{1} & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -8 & -7 \end{array}\right]$$

$$\xrightarrow{\substack{-2R_2+R_1 \\ 2R_2+R_3}} \left[\begin{array}{cccc|c} \underline{1} & 0 & 1 & -4 & -1 \\ 0 & \underline{1} & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_3/-1} \left[\begin{array}{cccc|c} \underline{1} & 0 & 1 & -4 & -1 \\ 0 & \underline{1} & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} \end{array}\right]$$

rank [A] = 2

rank [A|b] = 3

Since rank [A] < rank [A|b], then the system has *no solutions*.

EXAMPLE 1-7:

i. *System of equations:*

$$x_1 - x_5 = -20$$
$$-x_2 + x_4 = 10$$
$$-x_1 + x_2 - x_3 = 0$$
$$x_5 - x_6 - x_7 = 0$$
$$-x_4 + x_6 = -60$$

Constraints:

(1) x_i must be an integer (equivalent to $x_i \in Z$)

(2) $x_i \geq 0$

ii. General solution:

$$x_1 = -20 + s + t$$
$$x_2 = 50 + s$$
$$x_3 = 70 - t$$
$$x_4 = 60 + s$$
$$x_5 = s + t$$
$$x_6 = s$$
$$x_7 = t$$

iii. Close DE. Find minimum flow along AC.

If DE is closed, then $x_6 = s = 0$.

Then, the general solution becomes

$$x_1 = -20 + t$$
$$x_2 = 50$$
$$x_3 = 70 - t$$
$$x_4 = 60$$
$$x_5 = t$$
$$x_6 = 0$$
$$x_7 = t$$

We analyze each of the xi's individually to determine their minimum flows, and then we take the most restricted value of *t* as the minimum flow of *t*.

$x_1 : t \geq 20$

x_2 : no t to analyze

$x_3 : t \geq 0$

x_4 : no t to analyze

$x_5 : t \geq 0$

x_6 : no t to analyze

$x_7 : t \geq 0$

The most restricted minimum for *t* is $t \geq 20$. Subbing in $t = 20$ into AC, which is x_1, we get the minimum flow along AC, which is $x_1 = -20 + 20 = 0$.

1 Systems of Linear Equations: Your Turn!

Solutions:

a) *Matrix form:*

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 5 \\ 3 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{array}\right]$$

Gauss time!

$$\left[\begin{array}{cccc|c} \lfloor 1 & 0 & 1 & 2 & 5 \\ 3 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cccc|c} \lfloor 1 & 0 & 1 & 2 & 5 \\ 0 & \lfloor 1 & -3 & -6 & -11 \\ 0 & 0 & \lfloor 1 & 2 & 6 \end{array}\right]$$

$$\xrightarrow[3R_3+R_2]{-R_3+R_1} \left[\begin{array}{cccc|c} \lfloor 1 & 0 & 0 & \boxed{0} & -1 \\ 0 & \lfloor 1 & 0 & \boxed{0} & 7 \\ 0 & 0 & \lfloor 1 & \boxed{2} & 6 \end{array}\right]$$

Solution type: Infinite solutions. One free variable.

Information from the matrix:

$$\begin{aligned} x &= -1 \\ y &= 7 \\ z + 2w &= 6 \\ w &= t \end{aligned}$$

Rewrite in terms of free variables:

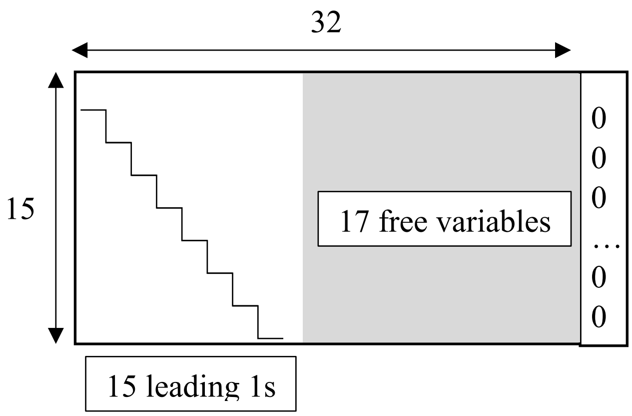
$$\begin{aligned} x &= -1 \\ y &= 7 \\ z &= 6 - 2t \\ w &= t \end{aligned}$$

parametric form

$$(x, y, z, w) = (-1, 7, 6, 0) + t(0, 0, -2, 1)$$

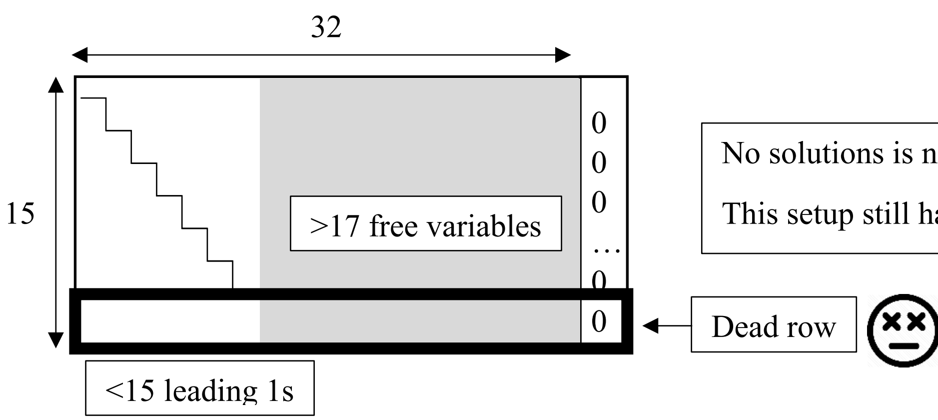
vector form

b) "Sketch" the matrix.



Unique solution is *impossible*.

Infinite solutions is *possible*.



No solutions is not *possible*.

This setup still has infinite solutions.

The only possibility is *infinite solutions.*

c)

Matrix form:

$$\left[\begin{array}{ccc|c} 3 & -2 & 9 & 14 \\ 1 & -1 & 2 & 4 \\ 2 & -4 & a & b \end{array}\right]$$

Gauss!

$$\left[\begin{array}{ccc|c} 3 & -2 & 9 & 14 \\ 1 & -1 & 2 & 4 \\ 2 & -4 & a & b \end{array}\right] \xrightarrow{\text{swap } R_1, R_2} \left[\begin{array}{ccc|c} \underline{1} & -1 & 2 & 4 \\ 3 & -2 & 9 & 14 \\ 2 & -4 & a & b \end{array}\right]$$

$$\xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} \underline{1} & -1 & 2 & 4 \\ 0 & \underline{1} & 3 & 2 \\ 0 & -2 & a-4 & b-8 \end{array}\right]$$

$$\xrightarrow[2R_2+R_3]{R_2+R_1} \left[\begin{array}{ccc|c} \underline{1} & 0 & 5 & 6 \\ 0 & \underline{1} & 3 & 2 \\ 0 & 0 & a+2 & b-4 \end{array}\right] \longleftarrow \boxed{\text{Stop here!}}$$

No solutions:

Need a zombie row. This happens if $a = -2, b \neq 4$.

Unique solution:

Need a non-zombie, non-dead row. This happens if $a \neq -2, b$ can be any real number..

Infinite solutions:

Need a dead row. This happens if $a = -2, b = 4$.

d) *Matrix form:*

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \\ 3 & 2 & 6 \end{array}\right]$$

Gauss time!

$$\left[\begin{array}{cc|c} \lfloor 1 & 1 & 3 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \\ 3 & 2 & 6 \end{array}\right] \xrightarrow[-3R_1+R_4]{\substack{-2R_1+R_2 \\ -2R_1+R_3}} \left[\begin{array}{cc|c} \lfloor 1 & 1 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & -1 & -3 \end{array}\right]$$

$$\xrightarrow{R_2/-1} \left[\begin{array}{cc|c} \lfloor 1 & 1 & 3 \\ 0 & \lfloor 1 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -3 \end{array}\right]$$

$$\xrightarrow[R_2+R_4]{-R_2+R_1} \left[\begin{array}{cc|c} \lfloor 1 & 0 & 0 \\ 0 & \lfloor 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Dead rows

rank [A] = 2

rank [A|b] = 2

columns (n) = 2

Since rank [A] = rank [A|b] = n, then the system has *a unique solution.*

e)

i. *System of equations:*

$$x_1 - x_5 = -50$$
$$-x_2 + x_3 + x_4 = 50$$
$$-x_1 + x_2 = 30$$
$$x_5 - x_6 - x_7 = 0$$
$$-x_4 + x_6 = -90$$

Constraints:

(1) x_i must be an integer (equivalent to $x_i \in I$)

(2) $x_i \geq 0$

ii. General solution:

$$x_1 = -50 + s + t$$
$$x_2 = -20 + s + t$$
$$x_3 = -60 + t$$
$$x_4 = 90 + s$$
$$x_5 = s + t$$
$$x_6 = s$$
$$x_7 = t$$

iii. Close DE. Find minimum flow along BC.

If DE is closed, then $x_6 = s = 0$.

Then, the general solution becomes

$$x_1 = -50 + t$$
$$x_2 = -20 + t$$
$$x_3 = -60 + t$$
$$x_4 = 90$$
$$x_5 = t$$
$$x_6 = 0$$
$$x_7 = t$$

All other flows must be ≥ 0. We can only change t. For all flows to be ≥ 0, $t \geq 60$ (because of x_3).
Then, the minimum flow along BC (x_2) is

$$x_2 = -20 + 60 = 40$$

EXAMPLE 2-8:

$$3A-2B=3\begin{bmatrix}1&5\\6&4\\2&3\\3&0\end{bmatrix}-2\begin{bmatrix}0&2\\3&3\\5&7\\1&2\end{bmatrix}$$
$$=\begin{bmatrix}3&15\\18&12\\6&9\\9&0\end{bmatrix}-\begin{bmatrix}0&4\\6&6\\10&14\\2&4\end{bmatrix}$$
$$=\begin{bmatrix}3&11\\12&6\\-4&-5\\7&-4\end{bmatrix}$$

$$2A^T-B^T=2\begin{bmatrix}1&5\\6&4\\2&3\\3&0\end{bmatrix}^T-\begin{bmatrix}0&2\\3&3\\5&7\\1&2\end{bmatrix}^T$$
$$=2\begin{bmatrix}1&6&2&3\\5&4&3&0\end{bmatrix}-\begin{bmatrix}0&3&5&1\\2&3&7&2\end{bmatrix}$$
$$=\begin{bmatrix}2&12&4&6\\10&8&6&0\end{bmatrix}-\begin{bmatrix}0&3&5&1\\2&3&7&2\end{bmatrix}$$
$$=\begin{bmatrix}2&9&-1&5\\8&5&-1&-2\end{bmatrix}$$

$$5\left(B^T\right)^T=5B$$
$$=5\begin{bmatrix}0&2\\3&3\\5&7\\1&2\end{bmatrix}$$
$$=\begin{bmatrix}0&10\\15&15\\25&35\\5&10\end{bmatrix}$$

EXAMPLE 2-9:

Combination	First Matrix		Second Matrix		Compatible?
	Rows	Columns	Rows	Columns	
AB	3	3	3	1	Yes
BA	3	1	3	3	No
AC	3	3	1	3	No
CA	1	3	3	3	Yes
BC	3	1	1	3	Yes
CB	1	3	3	1	Yes

$$AB = \begin{bmatrix} 20 \\ 6 \\ 32 \end{bmatrix}$$

$$CA = \begin{bmatrix} 7 & 11 & 9 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 2 & 4 \\ 6 & 6 & 12 \end{bmatrix}$$

$$CB = \begin{bmatrix} 16 \end{bmatrix}$$

<u>EXAMPLE 2-3:</u>

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A & 0 \\ I & B \end{bmatrix}$$

$$M^2 = \begin{bmatrix} A & 0 \\ I & B \end{bmatrix}\begin{bmatrix} A & 0 \\ I & B \end{bmatrix}$$

$$= \begin{bmatrix} A^2 & 0 \\ IA + BI & B^2 \end{bmatrix}$$

$$= \begin{bmatrix} A^2 & 0 \\ A + B & B^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$$

$$B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = B$$

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$M^2 = \begin{bmatrix} A^2 & 0 \\ A+B & B^2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ I & B \end{bmatrix} = M$$

$$M^3 = \begin{bmatrix} A & 0 \\ I & B \end{bmatrix}\begin{bmatrix} A & 0 \\ I & B \end{bmatrix} = \begin{bmatrix} A^2 & 0 \\ IA+BI & B^2 \end{bmatrix} = M$$

Therefore, $M^{2018} = M$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}^{2018} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

1.1 Matrix Algebra: Your Turn!

Solutions:

a)

$$5A+2B=5\begin{bmatrix}1&1&0\\2&5&1\\6&12&7\end{bmatrix}+2\begin{bmatrix}0&0&5\\5&1&9\\1&7&4\end{bmatrix}$$
$$=\begin{bmatrix}5&5&0\\10&25&5\\30&60&35\end{bmatrix}+\begin{bmatrix}0&0&10\\10&2&18\\2&14&8\end{bmatrix}$$
$$=\begin{bmatrix}5&5&10\\20&27&23\\32&74&43\end{bmatrix}$$

$$3B^T+4A^T=3\begin{bmatrix}0&0&5\\5&1&9\\1&7&4\end{bmatrix}^T+4\begin{bmatrix}1&1&0\\2&5&1\\6&12&7\end{bmatrix}^T$$
$$=3\begin{bmatrix}0&5&1\\0&1&7\\5&9&4\end{bmatrix}+4\begin{bmatrix}1&2&6\\1&5&12\\0&1&7\end{bmatrix}$$
$$=\begin{bmatrix}0&15&3\\0&3&21\\15&27&12\end{bmatrix}+\begin{bmatrix}4&8&24\\4&20&48\\0&4&28\end{bmatrix}$$
$$=\begin{bmatrix}4&23&27\\4&23&69\\15&31&40\end{bmatrix}$$

$$4\left(A^T\right)^T=4A$$
$$=4\begin{bmatrix}1&1&0\\2&5&1\\6&12&7\end{bmatrix}$$
$$=\begin{bmatrix}4&4&0\\8&20&4\\24&48&28\end{bmatrix}$$

b)

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}$$

$$M^2 = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A^2 & AB+BI \\ 0 & I^2 \end{bmatrix} = \begin{bmatrix} A^2 & AB+B \\ 0 & I \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$AB + B = B$$

$$M^2 = \begin{bmatrix} A^2 & AB+B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} = M$$

$$M^3 = M^2M = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} = M^2 = M$$

Therefore, $M^{2018} = M$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{2018} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.